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Department of Statistics
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STAT 3375Q: Introduction to Mathematical Statistics I

Spring 2024

Final Simulation Solutions Exam Date: 29 April 2024

Problem 1
Suppose that a random variable X can take each of the five values $-2, -1, 0, 1, 2$ with equal probability. Let $Y = |X| - X$.

a) Find the distribution Y. *(8 points)*

b) Compute the mean of Y. *(6 points)*

c) Compute the variance of Y. *(6 points)*

Solution:

a) • Given:

x	-2	-1	0	1	2
$p(x)$	1/5	1/5	1/5	1/5	1/5

• Computing the possible values of Y

$\|x\|$	2	1	0	1	2
$y = \|x\| - x$	4	2	0	0	0

• Therefore, Y has the following distribution:

y	4	2	0
$p(y)$	1/5	1/5	1/5 + 1/5 + 1/5 =3/5

b) $E(Y) = 4\left(\frac{1}{5}\right) + 2\left(\frac{1}{5}\right) + 0\left(\frac{3}{5}\right) = \frac{6}{5}$.

c) $E(Y^2) = 4^2\left(\frac{1}{5}\right) + 2^2\left(\frac{1}{5}\right) + 0^2\left(\frac{3}{5}\right) = \frac{16}{5} + \frac{4}{5} = 4$.
$V(Y) = E(Y^2) - \{E(Y)\}^2 = 4 - \left(\frac{6}{5}\right)^2 = 4 - \frac{36}{25} = \frac{100-36}{25} = \frac{64}{25}$.

□

Problem 2
Let X be a uniform RV over the interval $[-4, 6]$. Suppose we have another RV Y also uniform over the interval $[a, 4a]$.

a) Find the mean of X. *(5 points)*

b) Find $P(X \leq 2.4)$. *(5 points)*

c) Find $P(-3 \leq X - 2 \leq 3)$. *(5 points)*

d) Given that $P(X \leq 1) = P(Y \leq 1)$, find the value of a. *(5 points)*

Solution:

a) $E(X) = \frac{-4+6}{2} = 1$ mean of $X \sim \mathcal{U}(\theta_1, \theta_2)$: $E(X) = \frac{\theta_1+\theta_2}{2}$. Here, $\theta_1 = -4$ and $\theta_2 = 6$.

b)

$$\begin{aligned} P(X \leq 2.4) &= F(2.4) && \text{def'n of CDF.} \\ &= \frac{2.4 - (-4)}{6 - (-4)} && \text{CDF of } \mathcal{U}(\theta_1, \theta_2) : F(x) = \begin{cases} 0, & x < \theta_1 \\ \frac{x - \theta_1}{\theta_2 - \theta_1}, & \theta_1 \leq x \leq \theta_2 \\ 1, & x > \theta_2. \end{cases} \\ &= 0.64. \end{aligned}$$

c)

$$\begin{aligned} P(-3 \leq X - 2 \leq 3) &= P(-1 \leq X \leq 5) && \text{isolate } X \text{ by adding 5.} \\ &= F(5) - F(-1) && \text{probability = area under the curve.} \\ &= \frac{5 - (-4)}{6 - (-4)} - \frac{-1 - (-4)}{6 - (-4)} && \text{CDF of } \mathcal{U}(-4, 6). \\ &= \frac{9}{10} - \frac{3}{10} = 0.6. \end{aligned}$$

d)

$$\begin{aligned} P(X \leq 1) &= F(1) && \text{def'n of CDF.} \\ &= \frac{1 - (-4)}{6 - (-4)} && \text{CDF of } \mathcal{U}(-4, 6). \\ &= \frac{5}{10} = 0.5. \end{aligned}$$

$$\begin{aligned} P(Y \leq 1) &= F(1) && \text{def'n of CDF.} \\ &= \frac{1 - a}{4a - a} && \text{CDF of } \mathcal{U}(a, 4a). \\ &= \frac{1 - a}{3a}. \end{aligned}$$

From the given $P(X \leq 1) = P(Y \leq 1)$, this means that $0.5 = \frac{1-a}{3a}$.

$$\begin{aligned} 0.5 &= \frac{1 - a}{3a} \\ 1.5a &= 1 - a \\ 2.5a &= 1 \\ a &= 0.4. \end{aligned}$$

□

Problem 3
Let X and Y be continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 8xy, & \text{if } 0 < y < x, 0 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}.$$

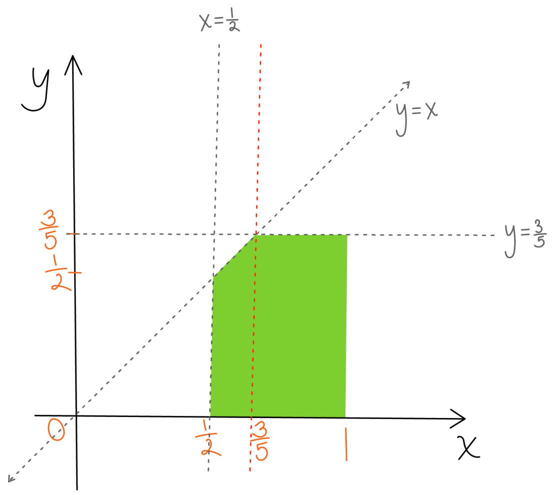
a) Find $P(X > 1/2)$. *(5 points)*

b) Find $P(Y < 3/5, X > 1/2)$. *(5 points)*

c) Find $P(Y < 3/5 | X > 1/2)$. *(5 points)*

d) Are X and Y independent? *(5 points)*

Solution:



a) To solve $P(X > 1/2)$, we need to find first the marginal PDF of X.

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^x 8xydy \\ &= 8x\frac{y^2}{2}\Big|_0^x = 4x^3, \quad 0 < x < 1. \end{aligned}$$

Now we can solve for $P(X > 1/2)$ as follows.

$$\begin{aligned} P(X > 1/2) &= \int_{1/2}^1 4x^3 dx \\ &= x^4\Big|_{1/2}^1 \\ &= 1^4 - \left(\frac{1}{2}\right)^4 \\ &= 1 - \frac{1}{16} = \frac{15}{16} = 0.9375. \end{aligned}$$

b)

$$
\begin{aligned}
P(Y < 3/5, X > 1/2) &= \int_{1/2}^{3/5}\int_0^x 8xydydx + \int_{3/5}^1\int_0^{3/5} 8xydydx \\
&= \int_{1/2}^{3/5} 8x\frac{y^2}{2}\Big|_0^x dx + \int_{3/5}^1 8x\frac{y^2}{2}\Big|_0^{3/5} dx \\
&= \int_{1/2}^{3/5} 4x^3dx + \int_{3/5}^1 \frac{36}{25}xdx \\
&= x^4\Big|_{1/2}^{3/5} + \frac{36}{25}\frac{x^2}{2}\Big|_{3/5}^1 \\
&= \left(\frac{3}{5}\right)^4 - \left(\frac{1}{2}\right)^4 + \frac{18}{25}\left(1 - \frac{9}{25}\right) \\
&= \frac{81}{625} - \frac{1}{16} + \frac{18}{25}\left(\frac{16}{25}\right) \\
&= \frac{81}{625} - \frac{1}{16} + \frac{288}{625} = \frac{369}{625} - \frac{1}{16} = \frac{5904 - 625}{10000} = \frac{5279}{10000} = 0.5279.
\end{aligned}
$$

c)

$$
\begin{aligned}
P(Y < 3/5 | X > 1/2) &= \frac{P(Y < 3/5, X > 1/2)}{P(X > 1/2)} \quad \text{def'n of conditional probability} \\
&= \frac{0.5279}{0.9375} \quad \text{answers in parts a and b} \\
&= 0.5631.
\end{aligned}
$$

d) For X and Y to be independent, we need to have $f(x, y) = f(x)f(y)$.

- From part a), the marginal PDF of X is

$$f(x) = 4x^3, \quad 0 < x < 1.$$

- Solve for the marginal PDF of Y, we get

$$
\begin{aligned}
f(y) &= \int_{-\infty}^{\infty} f(x, y)dx \\
&= \int_y^1 8xydx \\
&= 8y\frac{x^2}{2}\Big|_y^1 = 4y(1 - y^2), \quad 0 < y < 1.
\end{aligned}
$$

X and Y are NOT independent since $f(x)f(y) = (4x^3)\{4y(1 - y^2)\} = 16x^3y(1 - y^2)$ is not equal to $f(x, y) = 8xy$.

□

Problem 4

a) Let X be a Gaussian random variable with $\mu = 10$ and $\sigma^2 = 36$. Find $P(4 < X < 16)$. *(6 points)*

b) Let X be a Gaussian random variable with $\mu = 5$. If $P(X > 9) = 0.2$, compute $V(X)$. *(7 points)*

c) Let X be a Gaussian random variable with $\mu = 12$ and $\sigma^2 = 4$. Find the value of c such that $P(X > c) = 0.10$. *(7 points)*

Solution:

a)

$$\begin{aligned}
P(4 < X < 16) &= P\left(\frac{4-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{16-\mu}{\sigma}\right) \quad \text{standardization} \\
&= P\left(\frac{4-10}{6} \leq \frac{X-10}{6} \leq \frac{16-10}{6}\right) \\
&= P\left(-1 \leq Z \leq 1\right) \\
&= \Phi(1) - \Phi\left(-1\right) \quad \text{probability = area under the standard normal curve} \\
&= 0.84134 - 0.15866 \quad \text{Z-table values} \\
&= 0.6827.
\end{aligned}$$

b)

$$\begin{aligned}
P(X > 9) &= P\left(\frac{X-\mu}{\sigma} > \frac{9-\mu}{\sigma}\right) \quad \text{standardization} \\
&= P\left(\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right) \\
&= P\left(Z > \frac{4}{\sigma}\right) \\
&= 1 - P\left(Z \leq \frac{4}{\sigma}\right) \quad \text{complement} \\
&= 1 - \Phi\left(\frac{4}{\sigma}\right).
\end{aligned}$$

From the given, we want

$$\begin{aligned}
0.2 &= 1 - \Phi\left(\frac{4}{\sigma}\right) \\
\Phi\left(\frac{4}{\sigma}\right) &= 0.8
\end{aligned}$$

From the Z-table, $\Phi(0.84) = 0.8$. This means that

$$\begin{aligned}
\frac{4}{\sigma} &= 0.84 \\
\Rightarrow \sigma &= \frac{4}{0.84} \\
&= 4.76.
\end{aligned}$$

Therefore, $V(X) = \sigma^2 = 4.76^2 = 22.66$.

c)

$$\begin{aligned} P(X > c) &= P\left(\frac{X-\mu}{\sigma} > \frac{c-\mu}{\sigma}\right) && \text{standardization} \\ &= P\left(\frac{X-12}{2} > \frac{c-12}{2}\right) \\ &= P\left(Z > \frac{c-12}{2}\right) \\ &= 1 - P\left(Z \leq \frac{c-12}{2}\right) && \text{complement} \\ &= 1 - \Phi\left(\frac{c-12}{2}\right). \end{aligned}$$

From the given, we want

$$\begin{aligned} 0.1 &= 1 - \Phi\left(\frac{c-12}{2}\right) \\ \Phi\left(\frac{c-12}{2}\right) &= 0.9 \end{aligned}$$

From the Z-table, $\Phi(1.28) = 0.9$. This means that

$$\begin{aligned} \frac{c-12}{2} &= 1.28 \\ \Rightarrow c - 12 &= 2.56 \\ \Rightarrow c &= 14.56. \end{aligned}$$

□

Problem 5

Let X and Y be random variables such that

$$E(X)=2, \quad E(Y)=1, \quad E(X^2)=5, \quad E(Y^2)=10, \quad E(XY)=1.$$

a) Find $\text{Cov}(X,Y)$. *(4 points)*

b) Find $V(X)$. *(4 points)*

c) Find $V(Y)$. *(4 points)*

d) Find $\text{Corr}(X,Y)$. *(4 points)*

e) Find a number c so that X and $X+cY$ are uncorrelated. *(4 points)*

Solution:

a)

$$\begin{aligned}\text{Cov}(X,Y) &= E(XY)-E(X)E(Y)\\ &= 1-(2)(1)\\ &= -1.\end{aligned}$$

b)

$$\begin{aligned}V(X) &= E(X^2)-\{E(X)\}^2\\ &= 5-2^2\\ &= 1.\end{aligned}$$

c)

$$\begin{aligned}V(Y) &= E(Y^2)-\{E(Y)\}^2\\ &= 10-1^2\\ &= 9.\end{aligned}$$

d)

$$\begin{aligned}\text{Corr}(X,Y) &= \frac{\text{Cov}(X,Y)}{\sqrt{V(X)V(Y)}}\\ &= \frac{-1}{\sqrt{(1)(9)}}\\ &= -\frac{1}{3}.\end{aligned}$$

e) We want to find a c such that the covariance is zero. That is,

$$\begin{aligned}\text{Cov}(X,X+cY) &= E\{X(X+cY)\}-E(X)E(X+cY)\\ &= E(X^2+cXY)-E(X)\{E(X)+cE(Y)\}\\ &= E(X^2)+cE(XY)-\{E(X)\}^2-cE(X)E(Y)\\ &= 5+c(1)-2^2-c(2)(1)\\ &= 1-c.\end{aligned}$$

Solving c in the equation $1-c=0$, we have $c=1$. This means that X and $X+Y$ are uncorrelated.

□

Problem 6

Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\beta)$. Define $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ as the sample mean.

a) Find the distribution of the sample mean. *(8 points)*

b) Compute the mean of the sample mean. *(6 points)*

c) Compute the variance of the sample mean. *(6 points)*

Solution:

a)

$$
\begin{aligned}
m_{\overline{X}}(t) &= E\left(e^{t\overline{X}}\right) && \text{def'n of MGF} \\
&= E\left\{e^{t\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}\right\} && \text{given: } \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \\
&= E\left\{e^{\frac{t}{n}(X_1+X_2+\ldots+X_n)}\right\} \\
&= E\left(e^{\frac{t}{n}X_1} e^{\frac{t}{n}X_2} \cdots + e^{\frac{t}{n}X_n}\right) \\
&= E\left(e^{\frac{t}{n}X_1}\right) E\left(e^{\frac{t}{n}X_2}\right) \cdots E\left(e^{\frac{t}{n}X_n}\right) && \text{independence} \\
&= m_{X_1}\left(\frac{t}{n}\right) m_{X_2}\left(\frac{t}{n}\right) \cdots m_{X_n}\left(\frac{t}{n}\right) && \text{def'n of MGF} \\
&= \left(\frac{1}{1-\beta\frac{t}{n}}\right)\left(\frac{1}{1-\beta\frac{t}{n}}\right) \cdots \left(\frac{1}{1-\beta\frac{t}{n}}\right) && \text{MGF of Exp}(\beta)\text{ RV: } m(t) = \frac{1}{1-\beta t} \\
&= \left(\frac{1}{1-\beta\frac{t}{n}}\right)^{n} \\
&= \left(\frac{1}{1-\frac{\beta}{n}t}\right)^{n} && \text{isolate } t.
\end{aligned}
$$

The MGF above looks like the Gamma MGF: $m(t) = \frac{1}{(1-\beta t)^{\alpha}}$, when $\alpha = n$ and $\beta = \frac{\beta}{n}$.

Therefore, $\overline{X} \sim \text{Gamma}\left(n, \frac{\beta}{n}\right)$.

b)

$$
E(\overline{X}) = n\left(\frac{\beta}{n}\right) = \beta. \qquad \text{mean of Gamma RV is } \alpha\beta. \text{ Here, } \alpha = n \text{ and } \beta = \frac{\beta}{n}.
$$

c)

$$
V(\overline{X}) = n\left(\frac{\beta}{n}\right)^{2} = \frac{\beta^2}{n}. \qquad \text{mean of Gamma RV is } \alpha\beta^2. \text{ Here, } \alpha = n \text{ and } \beta = \frac{\beta}{n}.
$$

□

Problem 7
Let X have the following PDF:

$$f_X(x) = \begin{cases} \frac{x^2}{9}, & \text{if } 0 < x < 3; \\ 0, & \text{elsewhere.} \end{cases}$$

Find the PDF of $Y = X^3$ using the Jacobian method. *(20 points)*

Solution:

- Domain of X: $0 \leq x \leq 3$
- Codomain of Y: $0 \leq y \leq 27$
- Transformation: $h(x) = x^3$
- Inverse: Let $y = x^3$. To get the inverse, we need to solve for x. Solving for x, we have $x = y^{1/3}$. Therefore, $h^{-1}(y) = y^{1/3}$.
- Jacobian: $\frac{dh^{-1}(y)}{dy} = \frac{1}{3}y^{-2/3}$

$$\begin{aligned} f_Y(y) &= f_X\{h^{-1}(y)\}\left|\frac{dh^{-1}(y)}{dy}\right| \\ &= \frac{1}{9}(y^{1/3})^2\left|\frac{1}{3}y^{-2/3}\right| \\ &= \frac{1}{27}y^{2/3}y^{-2/3} \\ &= \frac{1}{27}. \end{aligned}$$

□