

STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 8: Continuous Random Variables

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Outline

- 1 Midterm 1 Solutions
- 2 Previously...
 - | Discrete Random Variables
 - | Special Discrete Distributions
- 3 The Cumulative Distribution Function
- 4 Continuous Random Variables

Midterm 1 Solutions

Problem 1

A pair of events A and B cannot be simultaneously mutually exclusive and independent. Prove that if $P(A) > 0$ and $P(B) > 0$, then:

- a) If A and B are mutually exclusive, they cannot be independent. (10 points)
- b) If A and B are independent, they cannot be mutually exclusive. (10 points)

Solution:

- a) Given: A and B are mutually exclusive.

) $A \cap B = \emptyset$.

) $P(A \cap B) = 0$.

If A and B are independent (p argument), then $P(A \cap B) = P(A)P(B) > 0$ (q argument) since $P(A) > 0$ and $P(B) > 0$. This contradicts $P(A \cap B) = 0$.

Since it is given that $P(A) > 0$ and $P(B) > 0$ (-q), then A and B cannot be independent (-p).

- b) Given: A and B are independent.

) $P(A \cap B) = P(A)P(B)$.

) $P(A \cap B) > 0$ since $P(A) > 0$ and $P(B) > 0$.

) $P(A \cap B) \neq 0$.

) A and B are not mutually exclusive.



Problem 2

Prove each of the following statements. (Assume that any conditioning event has positive probability.)

- a If $P(B) = 1$, then $P(A|B) = P(A)$ for any A . (5 points)
- b If $A \subseteq B$, then $P(B|A) = 1$ and $P(A|B) = \frac{P(A)}{P(B)}$. (5 points)
- c If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}. \quad (5 \text{ points})$$

- d $P(A \setminus B \setminus C) = P(A|B \setminus C)P(B|C)P(C)$. (5 points)

Solution:

- a $P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A \setminus B)$.

Let B and \bar{B} be a partition of the sample space S , i.e., $S = B \cup \bar{B}$ and $B \cap \bar{B} = \emptyset$.

By law of total probability, $P(A) = P(A \cap B) + P(A \cap \bar{B})$.

Since $(A \cap \bar{B}) \cap B = \emptyset$ and $P(\bar{B}) = 1 - P(B) = 1 - 1 = 0$, $P(A \cap \bar{B}) = 0$.

Hence, $P(A) = P(A \cap B)$.

Therefore, $P(A|B) = P(A)$.

Problem 2

Prove each of the following statements. (Assume that any conditioning event has positive probability.)

- a If $P(B) = 1$, then $P(A|B) = P(A)$ for any A . (5 points)
- b If $A \subseteq B$, then $P(B|A) = 1$ and $P(A|B) = \frac{P(A)}{P(B)}$. (5 points)
- c If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}. \quad (5 \text{ points})$$

- d $P(A \setminus B \setminus C) = P(A|B \setminus C)P(B|C)P(C)$. (5 points)

Solution:

- b If $A \subseteq B$, then $A \setminus B = \emptyset$.
) $P(B|A) = \frac{P(B \setminus A)}{P(A)} = \frac{P(\emptyset)}{P(A)} = 1$. Also, $P(A|B) = \frac{P(A \setminus B)}{P(B)} = \frac{P(\emptyset)}{P(B)}$.
- c If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

$$P(A|A \cup B) = \frac{P(A \setminus (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B)}.$$

- d $P(A \setminus B \setminus C) = P(A \setminus (B \setminus C)) = P(A|B \setminus C)P(B \setminus C) = P(A|B \setminus C)P(B|C)P(C)$.



Problem 3

Cards are dealt, one at a time, from a standard 52-card deck.

- a If the first 2 cards are both spades, what is the probability that the next 3 cards are also spades? (6 points)
- b If the first 3 cards are all spades, what is the probability that the next 2 cards are also spades? (7 points)
- c If the first 4 cards are all spades, what is the probability that the next card is also a spade? (7 points)

Solution:

- a Approach 1.

$$\begin{aligned} P(S_3 S_4 S_5 | S_1 S_2) &= \frac{P(S_3 \setminus S_4 \setminus S_5 \setminus S_1 \setminus S_2)}{P(S_1 \setminus S_2)} \\ &= \frac{\left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\left(\frac{10}{49}\right)\left(\frac{9}{48}\right)}{\left(\frac{13}{52}\right)\left(\frac{12}{51}\right)} = 0.084 \end{aligned}$$

Approach 2. 11 spades left, 50 cards left.

$$P(S_3 S_4 S_5 | S_1 S_2) = \frac{\binom{11}{3}}{\binom{50}{3}} = 0.084.$$

Problem 3

Cards are dealt, one at a time, from a standard 52-card deck.

- a If the first 2 cards are both spades, what is the probability that the next 3 cards are also spades? (6 points)
- b If the first 3 cards are all spades, what is the probability that the next 2 cards are also spades? (7 points)
- c If the first 4 cards are all spades, what is the probability that the next card is also a spade? (7 points)

Solution:

- b Approach 1.

$$\begin{aligned} P(S_4 S_5 / S_1 S_2 S_3) &= \frac{P(S_3 \setminus S_4 \setminus S_5 \setminus S_1 \setminus S_2)}{P(S_1 \setminus S_2 \setminus S_3)} \\ &= \frac{\left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\left(\frac{10}{49}\right)\left(\frac{9}{48}\right)}{\left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)} = 0.383 \end{aligned}$$

Approach 2. 10 spades left, 49 cards left.

$$P(S_4 S_5 / S_1 S_2 S_3) = \frac{\binom{10}{2}}{\binom{49}{2}} = 0.383.$$

Problem 3

Cards are dealt, one at a time, from a standard 52-card deck.

- a If the first 2 cards are both spades, what is the probability that the next 3 cards are also spades? (6 points)
- b If the first 3 cards are all spades, what is the probability that the next 2 cards are also spades? (7 points)
- c If the first 4 cards are all spades, what is the probability that the next card is also a spade? (7 points)

Solution:

- c Approach 1.

$$\begin{aligned} P(S_5 | S_1 S_2 S_3 S_4) &= \frac{P(S_3 \setminus S_4 \setminus S_5 \setminus S_1 \setminus S_2)}{P(S_1 \setminus S_2 \setminus S_3 \setminus S_4)} \\ &= \frac{\binom{13}{52} \binom{12}{51} \binom{11}{50} \binom{10}{49} \binom{9}{48}}{\binom{13}{52} \binom{12}{51} \binom{11}{50} \binom{10}{49}} = 0.1875 \end{aligned}$$

Approach 2. 9 spades left, 48 cards left.

$$P(S_5 | S_1 S_2 S_3 S_4) = \frac{\binom{9}{1}}{\binom{48}{1}} = 0.1875.$$



Problem 4

Suppose that 5% of men (M) and 0.25% of women (F) are color blind. A person is chosen at random and that person is color blind (CB). What is the probability that the person is male? (Assume males and females to be in equal numbers.) (20 points)

Solution:

Using Bayes' rule, we have

$$\begin{aligned}P(M|CB) &= \frac{P(CB|M)P(M)}{P(CB|M)P(M) + P(CB|F)P(F)} \\&= \frac{(0.05)(0.5)}{(0.05)(0.5) + (0.0025)(0.5)} \\&= 0.9524.\end{aligned}$$



Problem 5

Suppose X has the geometric PMF $p(x) = \frac{1}{3} \left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \dots$. Determine the probability distribution of $Y = \frac{X}{X+1}$. Note that both X and Y are discrete random variables. To specify the probability distribution means to specify its PMF. (20 points)

Solution:

$$\begin{aligned}P(Y = y) &= P\left(\frac{X}{X+1} = y\right) \\&= P(X = Xy + y) \\&= P(X - Xy = y) \\&= P(X(1 - y) = y) \\&= P\left(X = \frac{y}{1 - y}\right) \\&= p\left(\frac{y}{1 - y}\right) \\&= \frac{1}{3} \left(\frac{2}{3}\right)^{\frac{y}{1 - y}},\end{aligned}$$

where $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{x}{x+1}, \dots$



Problem 6

Suppose the random variable X has a Poisson distribution such that $P(X = 1) = P(X = 2)$.

- a Find $P(X = 4)$. (10 points)
- b Find $P(X \leq 4 | X \leq 2)$. (10 points)

Solution: Recall the Poisson PMF for a random variable Y : $p(y) = \frac{\lambda^y}{y!} e^{-\lambda}$.

a

$$\begin{aligned}P(X = 1) &= P(X = 2) \\p(1) &= p(2) \\ \frac{\lambda e^{-\lambda}}{1!} &= \frac{\lambda^2 e^{-\lambda}}{2!} \\ \lambda &= \frac{\lambda^2}{2} \\ \lambda &= 2.\end{aligned}$$

$$\text{Therefore, } P(X = 4) = p(4) = \frac{2^4 e^{-2}}{4!} = \frac{2}{3} e^{-2}.$$

Problem 6

Suppose the random variable X has a Poisson distribution such that $P(X = 1) = P(X = 2)$.

- a Find $P(X = 4)$. (10 points)
- b Find $P(X = 4 | X = 2)$. (10 points)

Solution: Recall the Poisson PMF for a random variable Y : $p(y) = \frac{\lambda^y}{y!} e^{-\lambda}$.

b

$$\begin{aligned}
 P(X = 4 | X = 2) &= \frac{P(X = 4 \text{ and } X = 2)}{P(X = 2)} = \frac{P(X = 4)}{P(X = 2)} \\
 &= \frac{1 \cdot [p(0) + p(1) + p(2) + p(3)]g}{1 \cdot [p(0) + p(1)]g} \\
 &= \frac{1 \cdot \left[\frac{e^{-2} 2^0}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} + \frac{2^3 e^{-2}}{3!} \right] g}{1 \cdot \left[\frac{e^{-2} 2^0}{0!} + \frac{2^1 e^{-2}}{1!} \right] g} \\
 &= \frac{1 \cdot \left[e^{-2} \left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} \right) \right] g}{1 \cdot \left[e^{-2} \left(\frac{2^0}{0!} + \frac{2^1}{1!} \right) \right] g} \\
 &= \frac{1 \cdot e^{-2} \left(1 + 2 + 2 + \frac{4}{3} \right)}{1 \cdot e^{-2} (1 + 2)} = \frac{1 \cdot e^{-2} \left(\frac{19}{3} \right)}{1 \cdot e^{-2} (3)} = 0.24.
 \end{aligned}$$



Problem 7

Let Y be the number of successes throughout n independent repetitions of a random experiment having the probability of success $p = \frac{1}{4}$. Determine the smallest value of n so that $P(Y \leq 1) \geq 0.70$. (20 points)

Solution:

Recall the binomial PMF for a random variable Y : $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$,
 $y = 0, 1, \dots, n$.

Note that it is easier to solve the probabilities of the complement.

$$\begin{aligned} P(Y \leq 1) &= 1 - P(Y = 0) \\ &= 1 - \binom{n}{0} \left(\frac{1}{4}\right)^0 \left(1 - \frac{1}{4}\right)^{n-0} \\ &= 1 - \left(\frac{3}{4}\right)^n. \end{aligned}$$

We need to find n such that

$$1 - \left(\frac{3}{4}\right)^n \geq 0.70 \iff \left(\frac{3}{4}\right)^n \leq 0.3 \iff n \log\left(\frac{3}{4}\right) \leq \log(0.3) \iff n > \frac{\log(0.3)}{\log\left(\frac{3}{4}\right)} = 4.19.$$

Therefore, n must be 5 so that $P(Y \leq 1) \geq 0.70$.



Previously...

Discrete Random Variables

- | **Random Variable:** is a **function** that maps outcomes into real numbers.
- | **Discrete Random Variables:** random variables whose values take only a **finite** or countably infinite number of possible values.
- | **Expected Value of Discrete Random Variable Y :**

$$\mu = E(Y) = \sum_y yp(y).$$

- | **Variance of Discrete Random Variable Y :**

$$\sigma^2 = V(Y) = E\{(Y - \mu)^2\} = \sum_y (y - \mu)^2 p(y) = E(Y^2) - \mu^2.$$

- | **Standard Deviation of Discrete Random Variable Y :**

$$\sigma = \sqrt{V(Y)}.$$

Special Discrete Distributions

	Bernoulli	Binomial	Poisson
Usage	success or failure?	count num. of success in n trials	count num. of times an event occurs w/ n specified time interval
Parameters	p (prob. success)	n (num. of trials), p (prob. success)	λ (average or expected rate of occurrence)
Notation	$Y \sim Be(p)$	$Y \sim B(n, p)$	$Y \sim Poi(\lambda)$
PMF	$p(y) = p^y(1-p)^{1-y}$ $y = 0, 1$	$p(y) = \binom{n}{y} p^y(1-p)^{n-y}$ $y = 0, 1, 2, \dots, n$	$p(y) = \frac{\lambda^y}{y!} e^{-\lambda}$ $y = 0, 1, 2, \dots$
Mean	p	np	λ
Variance	$p(1-p)$	$np(1-p)$	λ

(cont'd next slide...)

Special Discrete Distributions (cont'd)

	Geometric Type 1	Negative Binomial Type 1	Hypergeometric
Usage	count num. of trials to get first success	count num. of trials to get rth success	count num. of success w/o replacement
Parameters	p (prob. success)	r (num. of success) p (prob. success)	N (population size) r (num. of success in population) n (sample size)
Notation	$Y \sim \text{Geo}(p)$	$Y \sim \text{NB}(r, p)$	$Y \sim \text{Hyper}(N, r, n)$
PMF	$p(y) = (1 - p)^{y-1} p$ $y = 1, 2, 3, \dots$	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$ $y = r, r+1, r+2, \dots$	$p(y) = \frac{\binom{r}{y} \binom{N-r}{n-y}}{\binom{N}{n}}$ $y = 0, 1, 2, \dots, n,$ $y \leq r$ and $n-y \leq N-r$
Mean	$\frac{1}{p}$	$\frac{r}{p}$	$\frac{nr}{N}$
Variance	$\frac{1-p}{p^2}$	$\frac{r(1-p)}{p^2}$	$n \left(\frac{r}{N} \right) \left(\frac{N-r}{N} \right) \left(\frac{N-n}{N-1} \right)$

The Cumulative Distribution Function

The Cumulative Distribution Function

Definition 4.1: The Cumulative Distribution Function (CDF)

Let Y denote any random variable (discrete or continuous). The *cumulative distribution function* of Y , denoted by $F(y)$, is such that $F(y) = P(Y \leq y)$ for $-\infty < y < \infty$.

- | **cumulative**: increasing or growing by accumulation or successive additions
- | **CDF**: essentially tells us what is the **accumulated** probability from $-\infty$ up until the value y .
- | Even if Y may not take all real numbers, the CDF is defined for all $y \in \mathbb{R}$.

Note: The **nature** of the CDF of the random variable **determines** whether it is continuous or discrete.

The Cumulative Distribution Function

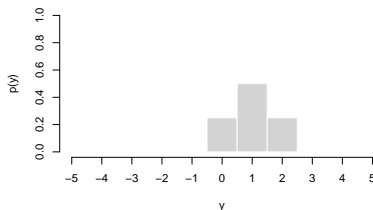
Example:

Suppose that $Y \sim B(n, p)$ with $n = 2$ and $p = 1/2$. Find $F(y)$.

Solution:

The PMF of Y is $p(y) = \binom{2}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{2-y}$, $y = 0, 1, 2$. $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$, $y = 0, 1, \dots, n$

y	$p(y)$
0	1/4
1	1/2
2	1/4



y	$F(y)$
-5	0
-0.5	0
0	1/4
0.5	1/4
0.9	1/4
1	$1/4 + 1/2 = 3/4$
1.9	$1/4 + 1/2 = 3/4$
2	$1/4 + 1/2 + 1/4 = 1$
3	$1/4 + 1/2 + 1/4 = 1$

The CDF of Y is $F(y) = P(Y \leq y) = \begin{cases} 0, & \text{for } y < 0, \\ 1/4, & \text{for } 0 \leq y < 1, \\ 3/4, & \text{for } 1 \leq y < 2, \\ 1, & \text{for } y \geq 2. \end{cases}$

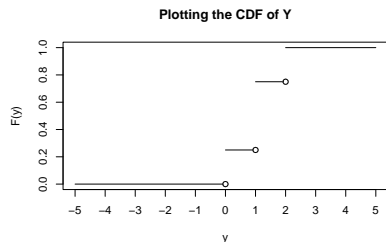
The Cumulative Distribution Function

Example:

Suppose that $Y \sim B(n, p)$ with $n = 2$ and $p = 1/2$. Find $F(y)$.

Solution:

$$\text{The CDF of } Y \text{ is } F(y) = P(Y \leq y) = \begin{cases} 0, & \text{for } y < 0, \\ 1/4, & \text{for } 0 \leq y < 1, \\ 3/4, & \text{for } 1 \leq y < 2, \\ 1, & \text{for } y \geq 2. \end{cases}$$



- | The graph of the CDF is a **step function**. The value of the CDF does not change between integers.
- | This is the case for all discrete random variables.
- | The value of the CDF **jumps** at the **possible values** of the random variable.
- | The **size of the jump** is given by the value of the PMF at that possible value of the random variable.

The Cumulative Distribution Function

Theorem 4.1: Properties of the CDF

If $F(y)$ is a distribution function, then

① $F(-\infty) = \lim_{y \rightarrow -\infty} F(y) = 0.$

② $F(\infty) = \lim_{y \rightarrow \infty} F(y) = 1.$

③ $F(y)$ is a nondecreasing function of y .

If y_1 and y_2 are any values such that $y_1 < y_2$, then $F(y_1) \leq F(y_2)$.

Proof:

① $F(-\infty) = P(Y \leq -\infty) = P(\emptyset) = 0.$

② $F(\infty) = P(Y \leq \infty) = P(S) = 1.$ S here is the sample space.

③ If $y_1 < y_2$, then the event $Y < y_1$ is a sub-set of the event $Y < y_2$.
Then by the monotonicity property of sets,

$$F(y_1) = P(Y < y_1) \leq P(Y < y_2) = F(y_2).$$

The Cumulative Distribution Function

Example:

Suppose that $Y \sim B(n, p)$ with $n = 2$ and $p = 1/2$. Check whether $F(y)$ satisfy the properties of CDF.

Solution:

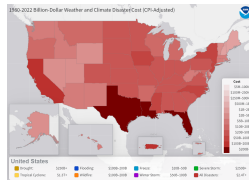
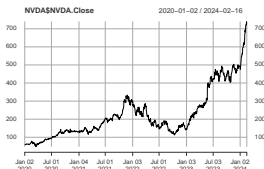
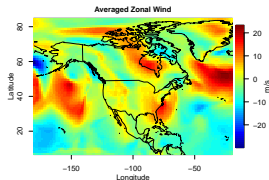
The CDF of Y is $F(y) = P(Y \leq y) = \begin{cases} 0, & \text{for } y < 0, \\ 1/4, & \text{for } 0 \leq y < 1, \\ 3/4, & \text{for } 1 \leq y < 2, \\ 1, & \text{for } y \geq 2. \end{cases}$

- 1 $F(-\infty) = 0$.
- 2 $F(\infty) = 1$.
- 3 The plot of $F(y)$ clearly shows that $F(y)$ is nondecreasing.

Continuous Random Variables

Continuous Random Variables

- | Not all random variables are discrete...
- | **Continuous random variables:** are random variables that can take on
 - | an **infinite** number of possible values.
 - | any value in an **interval**.
- | Continuous random variables can describe important real-world situations very well.



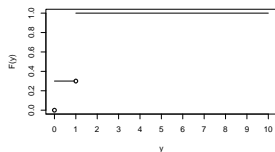
- | Continuous mathematics (Calculus) is frequently easier to work with than mathematics of discrete random variables and distributions.

Continuous Random Variables

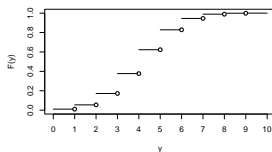
Definition 4.2: Continuous Random Variable

A random variable Y with CDF $F(y)$ is said to be *continuous* if $F(y)$ is continuous for $-\infty < y < \infty$.

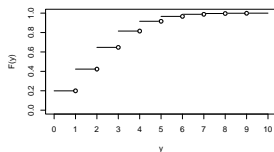
CDF of a Bernoulli RV



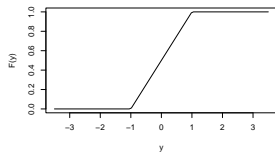
CDF of a Binomial RV



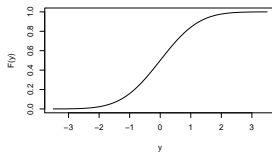
CDF of a Poisson RV



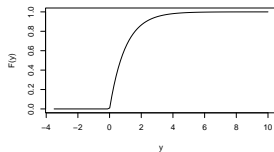
CDF of a Uniform RV



CDF of a Gaussian RV



CDF of an Exponential RV



Continuous Random Variables

Definition 4.3: CDF \rightarrow PDF (Differentiation)

Let $F(y)$ be the CDF for a continuous random variable Y . Then $f(y)$, given by

$$f(y) = \frac{dF(y)}{dy} = F'(y)$$

wherever the derivative exists, is called the *probability density function* (PDF) for the random variable Y .

Note: PDF \rightarrow CDF (Integration)

The CDF $F(y)$ can be retrieved from PDF $f(y)$ using the relationship:

$$F(y) = \int_1^y f(t)dt,$$

where t is used as the variable of integration.

Continuous Random Variables

Theorem 4.2: Properties of a PDF

If $f(y)$ is a PDF for a continuous random variable, then

- 1 $f(y) \geq 0$ for all y , $-\infty < y < \infty$.
- 2 $\int_{-\infty}^{\infty} f(y)dy = 1$.

Some Remarks:

- | PMF and PDF are **NOT** the same thing.
- | In the discrete case, to find $P(Y = y)$, we evaluate the PMF at a single point y .
- | The point mass for a **continuous** random variable Y is always equal to zero, i.e., $P(Y = y) = 0$.
- | In the continuous case, we **integrate the PDF** over a certain interval to find the **probability** that Y will fall in that interval.
- | $f(y)$ is **NOT** a probability.
- | $f(y)$ is the **height** of the PDF when the random variable Y takes on the value y .
- | $f(y)$ indicates how much probability is concentrated per unit length (dy) near y , or how dense the probability is near y .

Theorem 4.3

If the random variable Y has PDF $f(y)$ and $a < b$, then the probability that Y falls in the interval $[a, b]$ is

$$P(a \leq Y \leq b) = \int_a^b f(y) dy.$$

Note: The following four probabilities are equivalent for a continuous random variable Y :

$$P(a < Y < b) = P(a \leq Y < b) = P(a < Y \leq b) = P(a \leq Y \leq b).$$

This means that whether or not the endpoints of an interval are included, makes no difference when computing the probability of the interval.

Continuous Random Variables

Example 1:

Let X be a random variable with PDF $f(x) = 2x$ for $0 \leq x \leq 1$, and is 0 otherwise.

- a Find the CDF.
- b Find $P(\frac{1}{3} \leq X \leq \frac{1}{2})$.

Solution:

- a $F(x) = \int_0^x 2x dx = x^2, \quad 0 \leq x \leq 1.$
- b $P(\frac{1}{3} \leq X \leq \frac{1}{2}) = \int_{1/3}^{1/2} 2x dx = x^2 \Big|_{1/3}^{1/2} = (\frac{1}{2})^2 - (\frac{1}{3})^2 = \frac{9-4}{36} = \frac{5}{36}.$

Continuous Random Variables

Example 2:

The length of time to failure (in hundred of hours) for a transistor is a random variable Y with CDF

$$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y^2}, & y \geq 0. \end{cases}$$

- a Show that $F(y)$ has the properties of a CDF.
- b Find the PDF $f(y)$.
- c Find the probability that the transistor operates for at least 200 hours.

Solution:

- a The properties of a CDF are satisfied since:
 - | $\lim_{y \rightarrow -\infty} F(y) = \lim_{y \rightarrow -\infty} 0 = 0.$
 - | $\lim_{y \rightarrow \infty} F(y) = \lim_{y \rightarrow \infty} (1 - e^{-y^2}) = 1.$
 - | $F(y_2) - F(y_1) = e^{-y_1^2} - e^{-y_2^2} > 0$ if $y_1 < y_2.$

(exp. function w/ negative exponent is monotone decreasing)

Continuous Random Variables

Example 2:

The length of time to failure (in hundred of hours) for a transistor is a random variable Y with CDF

$$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y^2}, & y \geq 0. \end{cases}$$

- a Show that $F(y)$ has the properties of a CDF.
- b Find the PDF $f(y)$.
- c Find the probability that the transistor operates for at least 200 hours.

Solution:

- b Differentiating the CDF, we get the PDF

$$f(y) = F'(y) = \begin{cases} 0, & y < 0 \\ 2ye^{-y^2}, & y \geq 0. \end{cases}$$

Continuous Random Variables

Example 2:

The length of time to failure (in hundred of hours) for a transistor is a random variable Y with CDF

$$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y^2}, & y \geq 0. \end{cases}$$

- a Show that $F(y)$ has the properties of a CDF.
- b Find the PDF $f(y)$.
- c Find the probability that the transistor operates for at least 200 hours.

Solution:

- c The probability that the transistor operates for at least 200 hours is

$$P(Y \geq 2) = 1 - P(Y < 2) = 1 - F(2),$$

since F is continuous at $y = 2$ implies that $P(Y = 2) = 0$.

Therefore,

$$P(Y \geq 2) = 1 - F(2) = 1 - (1 - e^{-4}) = e^{-4}.$$

Discrete vs. Continuous Random Variables

	Discrete	Continuous
Probabilistic Question	What is the probability that Y is equal to y ? <i>We assign probabilities to individual outcomes.</i>	What is the probability of observing Y within the interval $[a, b]$? <i>We assign probabilities to intervals of outcomes.</i>
Values the r.v. can take	discrete points (countable)	continuous intervals (uncountable)
Probability Distribution	$p(y)$: probability mass function or PMF $0 \leq p(y) \leq 1$ $\sum_y p(y) = 1$	$f(y)$: probability density function or PDF $\frac{dF(y)}{dy} = f(y) \geq 0$ $\int_a^b f(y) dy = 1$
Probability =	value of PMF at y	area under the PDF, bounded by a and b
Mean	$\mu = E(Y) = \sum_y yp(y)$	$\mu = E(Y) = \int_a^b yp(y)$
Variance	$\sigma^2 = V(Y) = E\{(Y - \mu)^2\}$ $= \sum_y (y - \mu)^2 p(y)$	$\sigma^2 = V(Y) = E\{(Y - \mu)^2\}$ $= \int_a^b (y - \mu)^2 p(y)$

Questions?

Homework Exercises: 4.5, 4.9, 4.13, 4.15, 4.17

Solutions will be discussed this Friday by the TA.