STAT 3375Q: Introduction to Mathematical Statistics I

Review: Midterm 2

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Outline

Problem 1

Suppose X and Y are independent standard normal random variables. That is, $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$.

a. Find $\text{Cov}(X, Y)$.

Solution:

Since X and Y are independent, $\text{Cov}(X, Y) = 0$. □

Problem 1

Suppose X and Y are independent standard normal random variables. That is, $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$.

(b) Find $E(X^2Y^2)$.

Solution:

- Since X and Y are independent, we can split the expected value as follows:
$$E(X^2Y^2) = E(X^2)E(Y^2).$$
- Solving for $E(X^2)$, we have
$$\begin{aligned} E(X^2) &= V(X) + \{E(X)\}^2 && \text{def'n of variance} \\ &= 1 + 0^2 && X \sim \mathcal{N}(0,1) \\ &= 1. \end{aligned}$$
- Solving for $E(Y^2)$, we have
$$\begin{aligned} E(Y^2) &= V(Y) + \{E(Y)\}^2 && \text{def'n of variance} \\ &= 1 + 0^2 && Y \sim \mathcal{N}(0,1) \\ &= 1. \end{aligned}$$
- Therefore, $E(X^2Y^2) = E(X^2)E(Y^2) = (1)(1) = 1$.

Problem 1

Suppose X and Y are independent standard normal random variables. That is, $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$.

(c) Find $E(3X - 4Y)$.

Solution:

By the linearity of expectation, we have

$$\begin{aligned} E(3X-4Y) &= 3E(X) - 4E(Y) \\ &= 3(0) - 4(0) \quad \text{$X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$} \\ &= 0. \end{aligned}$$

□

Problem 1

Suppose X and Y are independent standard normal random variables. That is, $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$.

(d) Find $V(3X - 4Y)$.

Solution:

$$\begin{aligned} V(3X - 4Y) &= V(3X) + V(-4Y) \\ &\quad \text{Variance of the sum of independent RVs: } V(X+Y) = V(X) + V(Y) \\ &= 3^2 V(X) + (-4)^2 V(Y) \quad \text{Variance of a linear transform: } V(aX+b) = a^2 V(X) \\ &= 9(1) + 16(1) \quad X \sim \mathcal{N}(0,1) \text{ and } Y \sim \mathcal{N}(0,1) \\ &= 25. \end{aligned}$$

□

Problem 1

Suppose X and Y are independent standard normal random variables. That is, $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$.

(e) Find $P(-3 \leq 3X - 4Y \leq 5)$. Hint: Sum of 2 Gaussian RVs is a Gaussian RV.

Solution:

- Let $W = 3X - 4Y$.
- From part c) and d), we know that $W \sim \mathcal{N}(\mu = 0, \sigma^2 = 25)$.

$$
\begin{aligned}
P(-3 \leq W \leq 5) &= P\left(\frac{-3-\mu}{\sigma} \leq \frac{W-\mu}{\sigma} \leq \frac{5-\mu}{\sigma}\right) && \text{standardization} \\
&= P\left(\frac{-3-0}{\sqrt{25}} \leq \frac{W-0}{\sqrt{25}} \leq \frac{5-0}{\sqrt{25}}\right) \\
&= P\left(-\frac{3}{5} \leq Z \leq 1\right) \\
&= \Phi(1) - \Phi\left(-\frac{3}{5}\right) && \text{probability = area under the standard normal curve} \\
&= 0.84134 - 0.27425 && \text{Z-table values} \\
&= 0.5671. \quad \square
\end{aligned}
$$

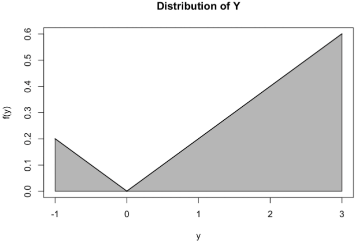
Problem 2

Consider a random variable Y with the PDF

$$f(y) = \frac{|y|}{5}, \quad -1 < y < 3.$$

Find $E(Y)$.

Solution:



$$f(y) = \begin{cases} -\frac{y}{5}, & -1 < y < 0 \\ \frac{y}{5}, & 0 \leq y < 3. \end{cases}$$

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y f(y) dy \quad \text{def'n of expected value} \\ &= \int_{-1}^{0} y \left(-\frac{y}{5}\right) dy + \int_{0}^{3} y \left(\frac{y}{5}\right) dy \\ &= \int_{-1}^{0} \left(-\frac{y^2}{5}\right) dy + \int_{0}^{3} \frac{y^2}{5} dy \\ &= \left(-\frac{y^3}{15}\right)\Big|_{-1}^{0} + \frac{y^3}{15}\Big|_{0}^{3} \\ &= -\frac{1}{15} + \frac{27}{15} = \frac{26}{15}. \quad \square \end{aligned}$$

Problem 3

Suppose that the completion time in hours T for the STAT 3375Q final exam follows a distribution with density

$$f(t) = \frac{2}{27}(t^2 + t), \quad 0 \leq t \leq 3.$$

What is the probability that a randomly chosen student finishes the exam during the second hour of the exam.

Solution:

$$P(1 < T < 2) = \int_1^2 \frac{2}{27}(t^2 + t)dt$$

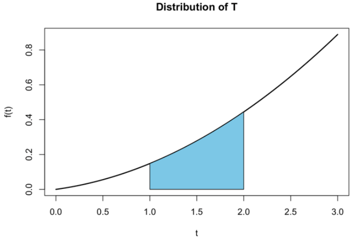
probability = area under density curve

$$= \frac{2}{27}\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Bigg|_1^2$$

$$= \frac{2}{27}\left(\frac{8}{3} + \frac{4}{2} - \frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{2}{27}\left(\frac{7}{3} + \frac{3}{2}\right) = \frac{2}{27}\left(\frac{14+9}{6}\right)$$

$$= \frac{23}{81}. \quad \square$$



Problem 4

Given that X has MGF

$$m(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t},$$

find $P(|X| \leq 1)$.

Solution:

Matching the MGF above to the MGF formula $m(t) = E(e^{tX}) = \sum_y e^{tx}p(x)$, we know that the given MGF corresponds to a discrete random variable with PMF:

$$p(x) = \begin{cases} \frac{1}{6}, & \text{if } x = -2, \\ \frac{1}{3}, & \text{if } x = -1, \\ \frac{1}{4}, & \text{if } x = 1, \\ \frac{1}{4}, & \text{if } x = 2. \end{cases}$$

Therefore,

$$P(|X| \leq 1) = P(X = -1) + P(X = 1) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}. \quad \square$$

Problem 5

Suppose X and Y are continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 15x^2y, & \text{if } 0 \leq x \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal PDF of X, $f(x)$.

Solution:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x,y)dy \quad \text{def'n of marginal PDF} \\ &= \int_{x}^{1} 15x^2 y dy \\ &= 15x^2 \frac{y^2}{2}\Big|_{x}^{1} \\ &= \frac{15}{2}x^2(1-x^2), \quad 0 \leq x \leq 1. \end{aligned}$$

Problem 5

Suppose X and Y are continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 15x^2y, & \text{if } 0 \le x \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(b) Find the conditional PDF of Y given X, $f(y|x)$.

Solution:

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f(x)} \quad \text{def'n of conditional PDF} \\ &= \frac{15x^2y}{\frac{15}{2}x^2(1-x^2)} \\ &= \frac{2y}{1-x^2}, \quad 0 \le x \le y \le 1. \end{aligned}$$

Problem 5

Suppose X and Y are continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 15x^2y, & \text{if } 0 \leq x \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

c. Find $P(Y \leq 1/2 | X = 1/4)$.

Solution:

$$\begin{aligned} P(Y \leq 1/2|X=1/4) &= \int_{1/4}^{1/2} f(y|x=1/4)dy \quad \text{conditional probability = area under the conditional PDF} \\ &= \int_{1/4}^{1/2} \frac{2y}{1-\left(\frac{1}{4}\right)^2}dy = \int_{1/4}^{1/2} \frac{2y}{1-\frac{1}{16}}dy \\ &\quad \text{Using the conditional PDF in part b) and fixing } x = \frac{1}{4} \\ &= \int_{1/4}^{1/2} \frac{2y}{\frac{15}{16}}dy = \frac{32}{15}\int_{1/4}^{1/2} ydy \\ &= \frac{32}{15}\left(\frac{y^2}{2}\right)\Big|_{1/4}^{1/2} = \frac{32}{15}\left(\frac{1}{8}-\frac{1}{32}\right) = \frac{32}{15}\left(\frac{3}{32}\right) = \frac{1}{5}. \quad \square \end{aligned}$$

Problem 5

Suppose X and Y are continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 15x^2y, & \text{if } 0 \le x \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

d. Find $E(Y|X=x)$.

Solution:

$$\begin{aligned} E(Y|X=x) &= \int_{-\infty}^{\infty} yf(y|x)dy \quad \text{def'n of conditional expectation} \\ &= \int_x^1 y\frac{2y}{1-x^2}dy = \frac{1}{1-x^2}\int_x^1 2y^2dy \\ &= \frac{1}{1-x^2}\left(\frac{2y^3}{3}\right)\Bigg|_x^1 \\ &= \frac{1}{1-x^2}\left(\frac{2}{3}-\frac{2x^3}{3}\right) \\ &= \frac{2}{3}\left(\frac{1-x^3}{1-x^2}\right). \end{aligned}$$

Problem 6

Let X be a random variable with MGF

$$m(t) = \left(1 - \frac{t}{2}\right)^{-2}, \quad |t| < 2.$$

(a) Find $E(X)$.

Solution:

$$\begin{aligned} m'(t) &= (-2)\left(1 - \frac{t}{2}\right)^{-3}\left(-\frac{1}{2}\right) \\ &= \left(1 - \frac{t}{2}\right)^{-3} \\ E(X) = m'(0) &= \left(1 - \frac{(0)}{2}\right)^{-3} = 1. \end{aligned}$$

□

Problem 6

Let X be a random variable with MGF

$$m(t) = \left(1 - \frac{t}{2}\right)^{-2}, \quad |t| < 2.$$

b. Find $E(X^2)$.

Solution:

$$\begin{aligned} m''(t) &= \frac{d}{dt}\left\{\left(1 - \frac{t}{2}\right)^{-3}\right\} \\ &= (-3)\left(1 - \frac{t}{2}\right)^{-4}\left(-\frac{1}{2}\right) = \frac{3}{2}\left(1 - \frac{t}{2}\right)^{-4} \\ E(X^2) = m''(0) &= \frac{3}{2}\left(1 - \frac{(0)}{2}\right)^{-4} = \frac{3}{2}. \end{aligned}$$

Problem 6

Let X be a random variable with MGF

$$m(t) = \left(1 - \frac{t}{2}\right)^{-2}, \quad |t| < 2.$$

- (c) Find $V(X)$.

Solution:

$$\begin{aligned} V(X) &= E(X^2) - \{E(X)\}^2 \quad \text{def'n of variance} \\ &= \frac{3}{2} - 1^2 = \frac{1}{2}. \end{aligned}$$

□

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad V(X) = 1, \quad E(Y) = 2, \quad V(Y) = 2, \quad \text{Cov}(X, Y) = 1.$$

a. Find $E(X + 2Y)$.

Solution:

$$\begin{aligned} E(X + 2Y) &= E(X) + 2E(Y) && \text{linearity of expectation} \\ &= 1 + 2(2) && \text{given} \\ &= 5. \end{aligned}$$

□

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad V(X) = 1, \quad E(Y) = 2, \quad V(Y) = 2, \quad \text{Cov}(X, Y) = 1.$$

(b) Find $E(XY)$.

Solution:

Recall the covariance formula: $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$. Therefore,

$$\begin{aligned} E(XY) &= \text{Cov}(X, Y) + E(X)E(Y) \\ &= 1 + (1)(2) \quad \text{given} \\ &= 3. \end{aligned}$$

□

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad V(X) = 1, \quad E(Y) = 2, \quad V(Y) = 2, \quad \text{Cov}(X, Y) = 1.$$

(c) Find $V(X - 2Y + 1)$.

Solution:

$$\begin{aligned}
V(X - 2Y + 1) &= V(X - 2Y) \quad \text{Variance of a linear transform: } V(aX + b) = a^2 V(X) \\
&= V(X) + (-2)^2 V(Y) + 2\text{Cov}(X, -2Y) \\
&\quad \text{Variance of the sum: } V(X + Y) = V(X) + V(Y) + 2\text{Cov}(X, Y) \\
&= V(X) + 4V(Y) + 2(-2)\text{Cov}(X, Y) \\
&\quad \text{Covariance of linear transform: } \text{Cov}(aX + b, cY + d) = ac\text{Cov}(X, Y). \text{ Here } a = 1, c = -2. \\
&= V(X) + 4V(Y) - 4\text{Cov}(X, Y) \\
&= 1 + (4)(2) - 4(1) \quad \text{given} \\
&= 5.
\end{aligned}$$

□

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad V(X) = 1, \quad E(Y) = 2, \quad V(Y) = 2, \quad \text{Cov}(X, Y) = 1.$$

(d) Find $\text{Corr}(X, Y)$.

Solution:

$$\begin{aligned} \text{Corr}(X, Y) &= \frac{\text{Cov}(X, Y)}{\sqrt{V(X)V(Y)}} && \text{correlation formula} \\ &= \frac{1}{\sqrt{(1)(2)}} && \text{given} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

□

Questions?