Solutions for Week4 Discussion Session

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3.37

The probability of getting exactly k successes in n independent Bernoulli trials (with the same success rate p) is given by pmf of binomial distribution.
a. Not a binomial random variable.
b. Not a binomial random variable.
c. Binomial random variable with n = 100, p = proportion of high school students who scored above 1026.
d. Not a binomial random variable (not discrete).
e. Not binomial, since the sample was not selected among all female HS grads.

3.43

A recent EPA report notes that 70% of the island residents of Puerto Rico have reduced their electricity usage sufficiently to qualify for discounted rates. So, here the success rate is 0.7. Let, Y be a random variable denotes the number of qualifying subscribers. Then, Y is binomial with n = 5 (as sample size is taken 5) and p = .7.
a. Hence, all five qualify for the favorable rates i.e. $P(Y = 5) = .7^5 = .1681$.
b. At least four qualify for the favorable rates means, $P(Y \geq 4) = P(Y = 4) + P(Y = 5) = 5(.7^4)(.3) + .7^5 = .3601 + .1681 = 0.5282$.

3.55

Y is a binomial random variable with $n > 2$ trials and success probability p. From the Theorem,

$$E(y) = \sum_y yp(y = y)$$

Then, for this case,

$$E(Y(y-1)(y-2)) = \sum_y y(y-1)(y-2)P(y = y)$$

So, Given that, $p(y=y)=\begin{pmatrix} n \\ y \end{pmatrix} p^{y}(1-p)^{n-y}$

$$\begin{aligned}
& E(y(y-1)(y-2))=\sum_{y=0}^{n} y(y-1)(y-2) \frac{n!}{y!(n-y)!} p^{y}(1-p)^{n-y} \\
& =\sum_{y=0}^{n} \frac{y(y-1)(y-2) n!}{y(y-1)(y-2)(y-3)!(n-y)!} p^{y}(1-p)^{n-y} \\
& =\sum_{y=3}^{n} \frac{n!}{(y-3)!(n-y)!} p^{y}(1-p)^{n-y} \\
& =\sum_{y=3}^{n} \frac{n(n-1)(n-2)(n-3)!}{(y-3)!(n-y)!} p^{y}(1-p)^{n-y}
\end{aligned}$$

As, $y=3(1) n$ is now our support, we take a transformation to make it from $O(1) n^{*}$.

So, taking $y-3=z$ Now z is starting from 0 to n^{*}

here, $\quad n^{*}=n-3$; as previously we have, $\quad 3 \leqslant y \leqslant n$

$$\begin{aligned}
& \Rightarrow \quad 3-3 \leq y-3 \leq n-3 \quad \text{(subtracting 3 from each)} \\
& \Rightarrow \quad 0 \leq z \leq n-3
\end{aligned}$$

Hence,

$$\begin{aligned}
& E\{y(y-1)(y-2)\}=\sum_{y-z=0}^{n-3} \frac{n(n-1)(n-2)(n-3)!p^{y}(1-p)^{n-y}}{(y-3)!(n-y)!} \\
& =\sum_{z=0}^{n-3} \frac{n(n-1)(n-2)(n-3)!}{z!(n-3-(y-3))!} p^{y-3} p^{3}(1-p)^{n-y} \\
& =\sum_{z=0}^{n-3} \frac{n(n-1)(n-2)(n-3)!}{z!(n-3-z)!} p^{z} p^{3}(1-p)^{n-3-(y-3)} \\
& =\sum_{z=0}^{n-3} \frac{n(n-1)(n-2)(n-3)!p^{z} p^{3}(1-p)^{n-3-z}}{z!(n-3-z)!} \\
& =n(n-1)(n-2) p^{3} \sum_{z=0}^{n-3} \frac{(n-3)!p^{z}(1-p)^{n-3-z}}{z!(n-3-z)!}
\end{aligned}$$

All the terms free from Z variable we can take outside of the sum.

So, you can see that in the sum the whole expression is a binomial distribution with $(n-3, p)$ ie z binomial And for the, z is binomial p mf so,

$$\sum_{z=0}^{n-3} p(z=z)=1$$

Hence, $E\{y(y-1)(y-3)\}=n(n-1)(n-2) p^{3}$ Now, $E\left(y^{3}\right)-3 E\left(y^{2}\right)+2 E(y)=p^{3} n(n-1)(n-2)$

$$\begin{aligned}
& \Rightarrow E\left(y^{3}\right)-3\left\{n p(1-p)+n^{2} p^{2}\right\}+2 n p=p^{3} n(n-1)(n-2) \\
& \Rightarrow E\left(y^{3}\right)=n p+3 p^{2} n(n-1)+p^{3} n(n-1)(n-2)
\end{aligned}$$

As, $v(y) = np(1-p)$ and, $E\left(y^2\right) - E^2(y) = v(y) = \text{np}(1-p)$.

$$\Rightarrow E\left(y^2\right) = np(1-p) + n^2p^2$$

3.57

Let, Y = Number of successful explorations, then 10 − Y is the number of unsuccessful explorations. Given that there are 10 explorations, and probability of successful exploration is 0.1 i.e. 1 successful exploration out of 10.
Hence, the cost C is given by C = 20,000 + 30,000Y + 15,000(10 − Y).
Therefore, E(C) = 20,000 + 30,000(1) + 15,000(10 − 1) = 185,000.

3.65

(a)The maximum likelihood estimator for p is $\frac{Y}{n}$ (note that Y is the binomial random variable, not a particular value of it).

$$\begin{aligned} E(\frac{Y}{n}) &= \frac{1}{n}E(Y); \text{as n is not a random variable it will come out of expectation} \\ &= \frac{1}{n}np; \text{for binomial distribution E(Y) = np} \\ &= p \end{aligned}$$

So, $\frac{Y}{n}$ is an unbiased estimator of p as $E(Y/n) = p$.
(b)

$$\begin{aligned} V(\frac{Y}{n}) &= \frac{1}{n^2}V(Y); \text{as n is a constant, } V(n) = n^2 \\ &= \frac{np(1-p)}{n^2}; \text{for binomial distribution V(Y) = np(1-p)} \\ &= \frac{p(1-p)}{n} \end{aligned}$$

Here, if n gets large then $V(Y/n) = \frac{p(1-p)}{n}$ will decrease to 0.