Solutions for Week8 Discussion Session

4.93

Let, Y is a random variable, denotes time between fatal airplane accidents. Given that, Y is exponential with $\beta = 44$ days.
(a) If one of the accidents occurred on July 1 of a randomly selected year in the study period, then the probability that another accident occurred that same month; i.e. we need to find $P(Y \leq 31) = \int_0^{31} \frac{1}{44} e^{-y/44} = F(31) = 1 - e^{-31/44} = .5057$

(b) By using the property of exponential distribution, we can say that $V(Y) = (44)^2 = 1936$.

4.95

Y be an exponentially distributed random variable with mean β. Let's define another random variable X in the following way: $X = k \quad \text{if} \quad k-1 \leq Y < k \quad \text{for} \quad k = 1, 2, ...$

(a) So, $P(X = k)$ for each k = 1,2,.. will be equal to $P(k-1 \leq Y < k)$.
$P(k-1 \leq Y < k) = F(k) - F(k-1) = (1 - e^{-k/\beta}) - (1 - e^{-(k-1)/\beta}) = e^{-(k-1)/\beta} - e^{-k/\beta}$
Hence, $P(X = k) = e^{-(k-1)/\beta} - e^{-k/\beta}$

(b) $P(X = k) = e^{-(k-1)/\beta} - e^{-k/\beta} = e^{-(k-1)/\beta}(1 - e^{-1/\beta}) = (e^{-1/\beta})^{k-1}(1 - e^{-1/\beta})$
Thus, X has a geometric distribution with $p = 1 - e^{-1/\beta}$.

4.101

Given that the magnitude of earthquakes striking the region, say a random variable Y has a gamma distribution with $\alpha = 0.8$ and $\beta = 2.4$.

(a) The mean magnitude of earthquakes striking the region, $E(Y) = 0.8 * 2.4 = 1.92$; using the property of gamma distribution.

(b) The probability that the magnitude an earthquake striking the region will exceed 3.0 on the Richter scale; $P(Y > 3) = 0.21036$. [You have to use a software to get the probability value.]

(c) In 4.88, the region of North America the magnitude of earthquakes follow an exponential distribution with mean 2.4. Then, $P(Y > 3) = \int_3^{\infty} \frac{1}{2.4} e^{\frac{-y}{2.4}} dy = e^(- 3/2.4) = .2865$.
Hence, the probability found in Ex. 4.88 (a) is larger. There is greater variability with the exponential distribution.

(d) The probability that an earthquake striking the regions will fall between 2.0 and 3.0 on the Richter scale

is:
$P(2Y3) = P(Y > 2) – P(Y > 3) = .33979 – .21036 = .12943$.[probabilities are calculated using R software]

4.103

Let R denote the radius of a crater. Given that, R is exponential with mean 10 feet. The area is $A =^2$. The mean and variance of the areas i.e. A, produced by these explosive devices are:
$E(A) = E(\pi R^2) = \pi E(R^2) = \pi(var(R) + E^2(R)) = \pi(100 + 10^2) = 200\pi$.[$var(R) = 10^2$; by property of exponential distribution.]
$Var(\pi R^2) = \pi^2 Var(R^2) = \pi^2[E(R^4) - E^2(R^2)] = \pi^2[240000 - (200^2)] = 200000\pi^2$
$E(R^4) = \int_0^\infty \frac{1}{10} r^4 e^{\frac{-r}{10}} dr = \frac{10^5}{10} \int_0^\infty \frac{r^{(5-1)} e^{\frac{-r}{10}}}{10^5} dr = 10^4 \Gamma(5) = 240000$

4.111

Given, Y has a gamma distribution with parameters α and β.
(a) For any 'a' positive or negative value such that $\alpha + a > 0$, we have to find $E(Y^a)$.
$E(Y^a) = \int_0^\infty y^a \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^{a+\alpha-1} e^{-y/\beta} dy = \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(a+\alpha)\beta^{a+\alpha} = \beta^a \frac{\Gamma(a+\alpha)}{\Gamma(\alpha)}$

(b) For the gamma function $\Gamma(t)$ to be valid, we require $t > 0$.

(c) Now using the result in part(a):
$E(Y^1) = \beta^1 \frac{\Gamma(1+\alpha)}{\Gamma(\alpha)} = \beta \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta$

(d) $E(Y^{1/2}) = \beta^{1/2} \frac{\Gamma(1/2+\alpha)}{\Gamma(\alpha)}$. Here, α has to be greater than 0.

(e) $E(1/Y) = E(Y^{-1}) = \beta^{-1} \frac{\Gamma(-1+\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha-1)}{\beta\Gamma(\alpha)} = \frac{1}{\beta(\alpha-1)}$; $\alpha > 1$

$E(1/\sqrt{Y}) = E(Y^{-1/2}) = \beta^{-1/2} \frac{\Gamma(\frac{-1}{2}+\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha-0.5)}{\beta^{0.5}\Gamma(\alpha)}$; $\alpha > 0.5$

$E(1/Y^2) = E(Y^{-2}) = \beta^{-2} \frac{\Gamma(-2+\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha-2)}{\beta^2\Gamma(\alpha)} = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}$; $\alpha > 2$.