STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 12: Special Continuous Distributions: Gamma, Exponential, χ^2

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Outline

1. Previously...
   - Uniform Distribution
   - Normal (Gaussian) Distribution
   - Standard Normal Distribution
2. Gamma Distribution
3. Exponential Distribution
4. Chi-square (χ^2) Distribution

Previously...

Uniform Distribution

- Notation: $Y \sim U(\theta_1, \theta_2)$
- Parameters: θ_1 (minimum), θ_2 (maximum)
- PDF: $f(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 \leq y \leq \theta_2, \\ 0, & \text{elsewhere} \end{cases}$
- CDF: $F(y) = \begin{cases} 0, & y < \theta_1 \\ \frac{y - \theta_1}{\theta_2 - \theta_1}, & \theta_1 \leq y \leq \theta_2 \\ 1, & y > \theta_2. \end{cases}$
- Mean or Expected Value: $\frac{\theta_1 + \theta_2}{2}$
- Variance: $\frac{(\theta_2 - \theta_1)^2}{12}$

Normal (Gaussian) Distribution

- Notation: $Y \sim \mathcal{N}(\mu, \sigma^2)$
- Parameters: μ (mean), σ (standard deviation)
- PDF: $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, \quad -\infty \leq y \leq \infty$
- CDF: $F(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$ (no explicit form)
- Mean or Expected Value: μ
- Variance: σ^2

Standard Normal Distribution

- Notation: $Z \sim \mathcal{N}(0, 1)$
- Parameters: $\mu = 0$ (mean), $\sigma = 1$ (standard deviation)
- PDF: $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty \leq z \leq \infty$
- CDF: $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ (no explicit form)
- Mean or Expected Value: 0
- Variance: 1

Gamma Distribution

Gamma Distribution

Definition 4.9: Gamma Distribution

A random variable Y is said to have a *gamma probability distribution* with parameters $\alpha > 0$ and $\beta > 0$ if and only if the density function of Y is

$$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\Gamma(\cdot)$ is the gamma function, i.e.,

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1}e^{-y}\,dy.$$

- Notation: $Y \sim \text{Gam}(\alpha, \beta)$, read as: "$Y$ is a gamma random variable with shape parameter α and scale parameter β."
- Except when $\alpha = 1$ (an exponential distribution), it is impossible to obtain areas under the Gamma PDF by direct integration.
- CDF: no explicit form

Gamma Distribution

Properties: The Gamma Function

The Gamma function is given by

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy,$$

and satisfies the following properties:

1. If $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
2. $\Gamma(n) = (n-1)!$ for each integer $n \geq 1$.
3. $\Gamma(1/2) = \sqrt{\pi}$.

Proof: Left as an exercise...

Gamma Distribution: Prove $f(y)$ is a Valid PDF

Proof:

$$
\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty x^{\alpha-1} e^{-x} dx && \text{Gamma function} \\
1 &= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x} dx && \text{divide both sides by } \Gamma(\alpha) \\
1 &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y/\beta} \frac{1}{\beta} dy && \text{change of variables } x = \frac{y}{\beta} \Rightarrow dx = \frac{1}{\beta} dy \\
1 &= \int_0^\infty \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy. && \text{Gamma PDF integrates to 1}
\end{aligned}
$$

Therefore, the Gamma PDF is a valid PDF.

□

Gamma Distribution: The *Waiting Time* Distribution

- Used to describe the time between independent events
  - α: number of independent events
  - β: the average time between events
  - $Y \sim Gam(\alpha, \beta)$: the waiting time until α events have occurred
- Used to model continuous random variables that are always positive and have skewed (one tail is longer than the other) distributions
  - rainfalls
  - insurance claims
  - age of cancer incidence
  - wait time and service time in transportation and service industries

- Goal: to improve accuracy of DoorDash's delivery estimates (ETAs)
- Problems:
  - Under-prediction: (a late delivery) results in a really bad ordering experience
  - Over-prediction: (giving a higher estimate) might result in consumers not placing an order or getting a delivery before they get home to receive it.

Gamma Distribution: Applications

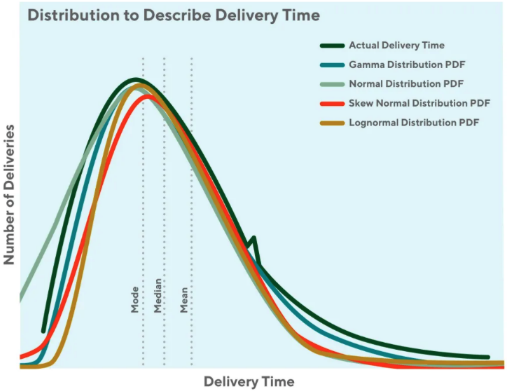


Figure 1. Comparison between actual delivery time distribution and commonly seen distributions.

Source: DoorDash

Distribution name	K-S test statistics
Normal	0.512
Skew normal (asymmetric)	0.784
Log-normal	0.999
Gamma	0.999

Table 1. K-S test results for different distributions toward the actual delivery time.

Source: DoorDash

- Problem: How to accurately predict ETA?
- Solution: Find the best distribution for the actual delivery times
- ETA prediction: mean of the best distribution

Gamma Distribution

Theorem 4.8: Gamma Distribution

If Y has a gamma distribution with parameters α and β, then

$$\mu = E(Y) = \alpha\beta \quad \text{and} \quad \sigma^2 = V(Y) = \alpha\beta^2.$$

Proof:

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} y f(y) dy && \text{def'n of expected value} \\
&= \int_0^{\infty} y \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy && f(y) = \begin{cases} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, & 0 \le y < \infty, \\ 0, & \text{elsewhere,} \end{cases} \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} y^{\alpha} e^{-y/\beta} dy \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} (\beta x)^{\alpha} e^{-x} (\beta dx) && \text{change of variables } x = \frac{y}{\beta} \Rightarrow dx = \frac{1}{\beta} dy \\
&= \frac{\beta^{\alpha+1}}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} x^{\alpha} e^{-x} dx \\
&= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1) && \text{Gamma function: } \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \\
&= \frac{\beta}{\Gamma(\alpha)} \alpha \Gamma(\alpha) && \text{HW Problem 4.81: } \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \\
&= \alpha\beta.
\end{aligned}$$

(cont'd next slide...)

Gamma Distribution

Proof:

$$
\begin{aligned}
E(Y^2) &= \int_{-\infty}^{\infty} y^2 f(y)dy \\
&= \int_0^{\infty} y^2 \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} y^{\alpha+1} e^{-y/\beta} dy \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} (\beta x)^{\alpha+1} e^{-x} (\beta dx) \\
&= \frac{\beta^{\alpha+2}}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} x^{\alpha+1} e^{-x} dx \\
&= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2) \\
&= \frac{\beta^2}{\Gamma(\alpha)} (\alpha+1)\alpha\Gamma(\alpha) \\
&= (\alpha+1)\alpha\beta^2.
\end{aligned}
$$

def'n of expected value

$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$

change of variables $x = \frac{y}{\beta} \Rightarrow dx = \frac{1}{\beta} dy$

Gamma function: $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$

$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1)$

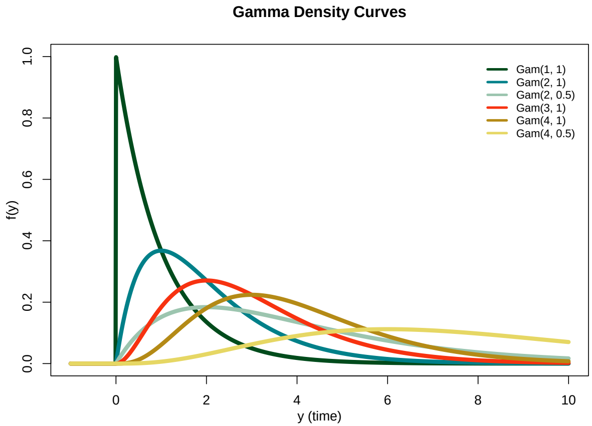
$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \Rightarrow \underline{\Gamma(\alpha+2) = (\alpha+1)\alpha\Gamma(\alpha)}$

$$
\begin{aligned}
V(Y) &= E(Y^2) - \{E(Y)\}^2 \\
&= (\alpha+1)\alpha\beta^2 - (\alpha\beta)^2 \\
&= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 \\
&= \alpha\beta^2.
\end{aligned}
$$

def'n of variance

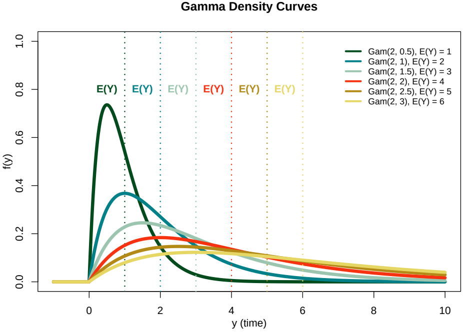
□

Gamma Distribution



- The distribution is asymmetrical and skewed to the right.
- The shape parameter α dictates the shape of the distribution.
- The scale parameter β dictates the spread of the distribution.
- As $\alpha \to \infty$, $\text{Gam}(\alpha, \beta) \to \mathcal{N}(\alpha\beta, \alpha\beta^2)$.

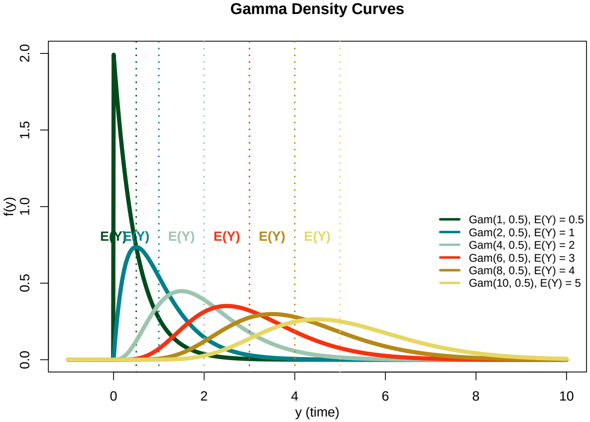
Gamma Distribution: The Effect of β



- rate $= \frac{1}{\beta}$ is the number of events per unit time
- Example:
  - $\beta = 0.5 \Rightarrow$ rate $= 2$ $\Rightarrow$ 2 deliveries every hour
  - $\beta = 1 \Rightarrow$ rate $= 1$ $\Rightarrow$ 1 delivery every hour
  - $\beta = 2 \Rightarrow$ rate $= 0.5$ $\Rightarrow$ 0.5 delivery every hour

- If $\alpha = 2$ and $\beta = 0.5$, $E(Y) = (2)(0.5) = 1$ $\Rightarrow$ expected waiting time is 1 hour for 2 deliveries
- If $\alpha = 2$ and $\beta = 2$, $E(Y) = (2)(2) = 4$ $\Rightarrow$ expected waiting time is 4 hours for 2 deliveries

Gamma Distribution: The Effect of α



- If $\alpha = 1$ and $\beta = 0.5$, $E(Y) = (1)(0.5) = 0.5$
  $\Rightarrow$ expected waiting time is 30 mins for 1 delivery
- If $\alpha = 2$ and $\beta = 0.5$, $E(Y) = (2)(0.5) = 1$
  $\Rightarrow$ expected waiting time is 1 hour for 2 deliveries

Gamma Distribution

Example 1:

Suppose that a random variable Y has PDF given BY

$$f(y) = \begin{cases} ky^3 e^{-y/2}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is the value of k that will make $f(y)$ a valid PDF?

Solution:

- Try matching the function above with the Gamma PDF:

$$f(y) = \begin{cases} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

- $\Rightarrow \beta = 2, \quad \alpha = 4, \quad k = \frac{1}{\beta^\alpha \Gamma(\alpha)} = \frac{1}{2^4 \Gamma(4)} = \frac{1}{96} = 0.01.$

□

Gamma Distribution

Example 1:

Suppose that a random variable Y has PDF given by

$$f(y) = \begin{cases} ky^3 e^{-y/2}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

b What is $E(Y^2)$?

Solution:

$$\begin{aligned} E(Y^2) &= V(Y) + \{E(Y)\}^2 && \text{variance formula: } V(Y) = E(Y^2) - \{E(Y)\}^2 \\ &= \alpha\beta^2 + (\alpha\beta)^2 && \text{mean and variance of Gamma RV} \\ &= (4)(2)^2 + \{(4)(2)\}^2 && \text{from part (a): } \alpha = 4, \beta = 2 \\ &= 80. \end{aligned}$$

□

Example 2:

Suppose that the time spent online to do homework by a randomly selected student has a Gamma distribution with mean 20 minutes and variance 80 minutes^2. What are the values of α and β?

Solution:

- mean $= \alpha\beta = 20 \Rightarrow \alpha = \frac{20}{\beta}$
- variance $= \alpha\beta^2 = 80 \Rightarrow \left(\frac{20}{\beta}\right)\beta^2 = 80 \Rightarrow \beta = 4 \Rightarrow \alpha = 5.$

□

Exponential Distribution

Exponential Distribution

Definition: Exponential Distribution

A random variable Y is said to have a *exponential probability distribution* with parameter $\beta > 0$ if and only if the density function of Y is

$$f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta}, & 0 \le y < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

- Notation: $Y \sim \text{Exp}(\beta)$, read as: "Y is an exponential random variable with parameter β."
- special case of Gamma distribution with $\alpha = 1$

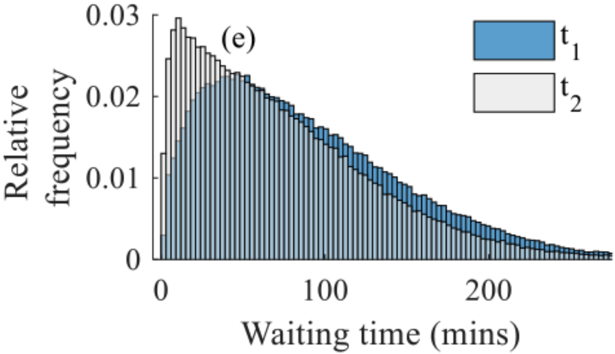
  Recall Gamma PDF: $f(y) = \begin{cases} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, & 0 \le y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$

- CDF: $F(y) = P(Y \le y) = \int_{-\infty}^{y} f(t)dt = \begin{cases} 0, & y < 0 \\ 1 - e^{-y/\beta}, & 0 \le y < \infty \end{cases}$

Exponential Distribution: Applications

Waiting Times in an Emergency Department

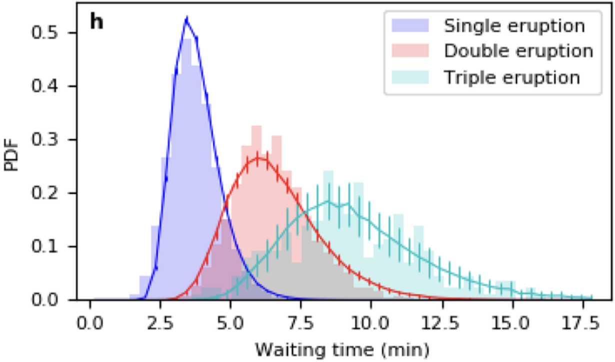
(t_1: from registration; t_2: from initial assessment)



Source: https://arxiv.org/pdf/2006.00335.pdf

Exponential Distribution: Applications

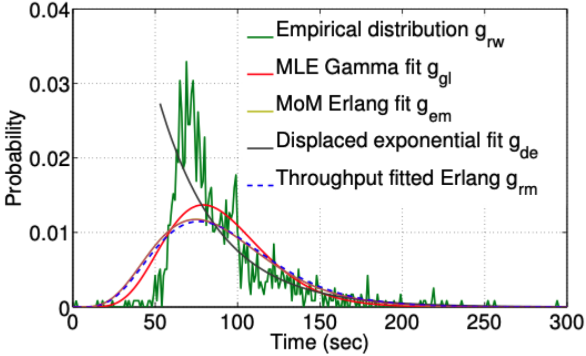
Geyser Eruptions Waiting Times



Source: Eruption Interval Monitoring at Strokkur Geyser, Iceland
https://doi.org/10.1029/2019GL085266

Exponential Distribution: Applications

Runway Service Times at Boston Logan Int'l Airport



Source: http://hdl.handle.net/1721.1/81186

Exponential Distribution

Theorem: Exponential Distribution

If Y has an exponential distribution with parameter β, then

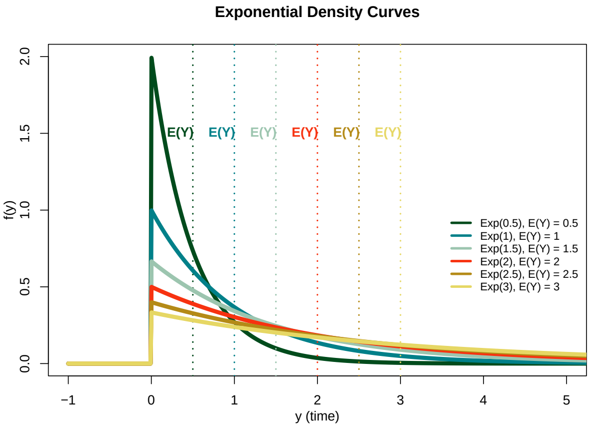
$$\mu = E(Y) = \beta \quad \text{and} \quad \sigma^2 = V(Y) = \beta^2.$$

Proof:

Since Y is also a gamma random variable with $\alpha = 1$, it follows that $\mu = E(Y) = \alpha\beta = \beta$ and $\sigma^2 = V(Y) = \alpha\beta^2 = \beta^2$.

□

Exponential Distribution: The Effect of β



- If $\beta = 0.5$, $E(Y) = 0.5$,
  $\Rightarrow$ expected waiting time is 30 mins for the event to happen
- If $\beta = 1$, $E(Y) = 1$,
  $\Rightarrow$ expected waiting time is 1 hour for the event to happen

Exponential Distribution

Memoryless Property

Suppose $Y \sim \text{Exp}(\beta)$. If $a > 0$ and $b > 0$, then

$$P(Y > a + b | Y > a) = P(Y > b).$$

Proof:

$$\begin{aligned}
P(Y > a + b | Y > a) &= \frac{P\{(Y > a + b) \cap (Y > a)\}}{P(Y > a)} && \text{def'n of conditional prob.} \\
&= \frac{P(Y > a + b)}{P(Y > a)} \\
&= \frac{1 - P(Y \leq a + b)}{1 - P(Y \leq a)} && \text{complement} \\
&= \frac{1 - \{1 - e^{-(a+b)/\beta}\}}{1 - (1 - e^{-a/\beta})} && \text{CDF: } F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y/\beta}, & 0 \leq y < \infty \end{cases} \\
&= \frac{e^{-(a+b)/\beta}}{e^{-a/\beta}} = e^{-a/\beta - b/\beta + a/\beta} = e^{-b/\beta}. \\
P(Y > b) &= 1 - P(Y \leq b) && \text{complement} \\
&= 1 - (1 - e^{-b/\beta}) = e^{-b/\beta}. \quad \square
\end{aligned}$$

Exponential Distribution

Example 3:

Ben is running late for his 9:00 am class. Suppose his possible arrival time can be modeled by an exponential random variable Y (in minutes after 9:00) with parameter $\beta = 15$. What is the probability that Ben arrives after 9:20?

Solution:

We are looking for $P(Y > 20)$.

Recall the exponential PDF: $f(y) = \frac{1}{\beta} e^{-y/\beta}$ for $y \geq 0$.

$$
\begin{aligned}
P(Y > 20) &= \int_{20}^{\infty} f(y) dy \\
&= \int_{20}^{\infty} \frac{1}{15} e^{-y/15} dy \\
&= (-e^{-y/15})\Big|_{20}^{\infty} \\
&= e^{-20/15} = 0.2636.
\end{aligned}
$$

Exponential Distribution

Example 4:

The time T required to repair a machine is exponentially distributed with mean 0.5.

a. What is the probability that a repair time exceeds 1/2 hour?

Solution:

Given: mean $= 0.5 \Rightarrow \beta = 0.5 \Rightarrow T \sim \text{Exp}(0.5)$

Recall the exponential PDF: $f(y) = \frac{1}{\beta} e^{-y/\beta}$ for $y \geq 0$.

$$
\begin{aligned}
P(T > 1/2) &= \int_{1/2}^{\infty} f(y) dy \\
&= \int_{1/2}^{\infty} \frac{1}{0.5} e^{-y/0.5} dy \\
&= \left(-e^{-y/0.5}\right)\Big|_{1/2}^{\infty} \\
&= e^{-1} = 0.3679.
\end{aligned}
$$

Example 4:

The time T required to repair a machine is exponentially distributed with mean with mean 0.5.

(b) What is the probability that a repair time exceeds 12.5 hours given that it is greater than 12?

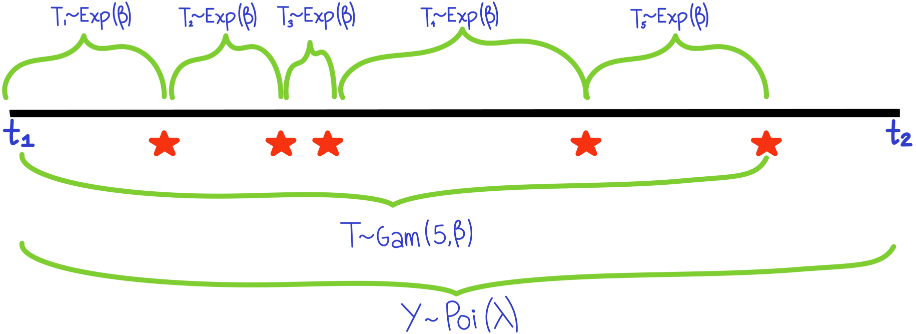
Solution:

Recall the memoryless property: $P(Y > a + b | Y > a) = P(Y > b)$

$$\begin{aligned} P(T > 12.5 | T > 12) &= P(T > 0.5) \quad \text{memoryless property: } a + b = 12.5, a = 12 \\ &= e^{-1} = 0.3679. \quad \text{from part (a)} \end{aligned}$$

□

Exponential-Gamma-Poisson Relationship



- T_i, $i = 1, 2, \ldots, 5$, is the waiting time until the next event
- T is the waiting time to the 5th event
  - $T = T_1 + T_2 + T_3 + T_4 + T_5$
- Y is the number of events between t_1 and t_2

Example 5:

A light bulb fails randomly with an expected lifetime of 20 days and is replaced immediately. Assume that this light bulb's lifetime has an exponential distribution.

a. What is the probability that the light bulb lasts longer than its expected lifetime?

Solution:

Let T be the lifetime of the light bulb.

Given: expected lifetime = 20 days = mean = $\beta \Rightarrow T \sim \text{Exp}(20)$

Recall the exponential PDF: $f(y) = \frac{1}{\beta} e^{-y/\beta}$ for $y \geq 0$.

$$\begin{aligned} P(T > 20) &= \int_{20}^{\infty} f(y) dy \\ &= \int_{20}^{\infty} \frac{1}{20} e^{-y/20} dy \\ &= \left(-e^{-y/20}\right)\Big|_{20}^{\infty} = e^{-1} = 0.3679. \end{aligned}$$

Example 5:

A light bulb fails randomly with an expected lifetime of 20 days and is replaced immediately. Assume that this light bulb's lifetime has an exponential distribution.

- (b) If the light bulb was installed 10 days ago, what is the probability that its lifetime (since installment day) will exceed the expected lifetime of 20 days?

Solution:

Recall the memoryless property: $P(Y > a + b | Y > a) = P(Y > b)$

$$\begin{aligned} P(T > 20 | T > 10) &= P(T > 10) \quad \text{memoryless property: } a + b = 20, a = 10 \\ &= \int_{10}^{\infty} f(y) dy \\ &= \int_{10}^{\infty} \frac{1}{20} e^{-y/20} dy \\ &= \left(-e^{-y/20}\right)\Big|_{10}^{\infty} = e^{-0.5} = 0.6065. \end{aligned}$$

Example 5:

A light bulb fails randomly with an expected lifetime of 20 days and is replaced immediately. Assume that this light bulb's lifetime has an exponential distribution.

- c) If you were to test 10 of these light bulbs, what is the probability that more than half will exceed the expected lifetime?

Solution:

- This is a binomial experiment with $n = 10$ and probability of success $p = P(T > 20) = 0.3679$ from part a).
- Let Y be the number of light bulbs exceeding the expected lifetime.
- $Y \sim B(10, 0.3679)$.

$$\begin{aligned} P(Y > 5) &= P(Y = 6) + \ldots + P(Y = 10) \\ &= p(6) + \ldots + p(10) \quad \text{PMF of binom: } p(y) = \binom{n}{y} p^y (1-p)^{n-y} \\ &= 0.0831 + 0.0276 + 0.0060 + 0.0008 + 4.5 \times 10^{-5} \\ &= 0.1175. \quad \square \end{aligned}$$

Example 5:

A light bulb fails randomly with an expected lifetime of 20 days and is replaced immediately. Assume that this light bulb's lifetime has an exponential distribution.

(d) If you have 2 spare bulbs, what is the probability that these (including the one currently in use) will be sufficient for the next 60 days?

Solution:

- This is a Poisson experiment with rate of occurrence, $\lambda = 3$ (num. of events per unit time).
  Note: 1 bulb failure per 20 days on average $\Rightarrow \frac{1}{20} = 0.05$ failures per day $\Rightarrow 0.05 \times 60 = 3$ failures in 60 days
- 2 spare light bulbs will be sufficient for 60 days if the number of bulb failures will not exceed 3 (total light bulbs available)
- Let X be the number of bulb failures. $X \sim \text{Poi}(\lambda = 3)$.

$$\begin{aligned} P(X \leq 3) &= P(X=0) + P(X=1) + P(X=2) + P(X=3) \\ &= p(0) + p(1) + p(2) + p(3) \quad \text{PMF of Poisson: } p(y) = \frac{\lambda^y}{y!} e^{-\lambda} \\ &= 0.0498 + 0.1494 + 0.2240 + 0.2240 = 0.6472. \quad \square \end{aligned}$$

Chi-square (χ^2) Distribution

Chi-square (χ^2) Distribution

Definition: Chi-square (χ^2) Distribution

Let $Z_1, Z_2, \ldots, Z_\nu$ be independent standard normal random variables, then the random variable

$$Y = \sum_{i=1}^{\nu} Z_i^2$$

has a *chi-square (χ^2) probability distribution* with $\nu > 0$ *degrees of freedom* and its density function is

$$f(y) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} y^{\nu/2-1} e^{-y/2}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

- Notation: $Y \sim \chi^2(\nu)$, read as: "Y is a χ^2 random variable with nu (ν) degrees of freedom."
- special case of Gamma distribution with $\alpha = \frac{\nu}{2}$ and $\beta = 2$
- CDF: no explicit form

χ^2 Distribution

Theorem: χ^2 Distribution

If Y has a χ^2 distribution with ν degrees of freedom, then

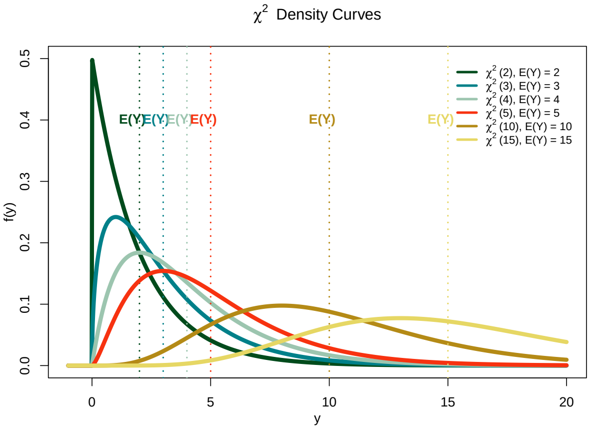
$$\mu = E(Y) = \nu \quad \text{and} \quad \sigma^2 = V(Y) = 2\nu.$$

Proof:

Since Y is also a gamma random variable with $\alpha = \frac{\nu}{2}$ and $\beta = 2$, it follows that $\mu = E(Y) = \alpha\beta = \left(\frac{\nu}{2}\right)(2) = \nu$ and
$\sigma^2 = V(Y) = \alpha\beta^2 = \left(\frac{\nu}{2}\right)(2^2) = 2\nu.$

□

χ^2 Distribution: The Effect of ν



- The curve is asymmetrical and skewed to the right.
- The degrees of freedom dictate the shape of the curve.
- As $\nu \to \infty$, $\chi(\nu) \to \mathcal{N}(\nu, 2\nu)$.
- Used for hypothesis testing

Questions?

Homework Exercises: 4.93, 4.95, 4.101, 4.103, 4.111

Solutions will be discussed this Friday by the TA.