STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 13: Special Continuous Distributions: Beta, Student's t

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March 6, 2024

Outline

1. Previously...
   - Gamma Distribution
   - Exponential Distribution
   - χ^2 Distribution

2. Beta Distribution

3. Student's t Distribution

Previously...

Gamma Distribution

- Notation: $Y \sim \text{Gam}(\alpha, \beta)$
- Parameters: $\alpha > 0$ (shape), $\beta > 0$ (scale)
- PDF: $f(y) = \begin{cases} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$

  where $\Gamma(\cdot)$ is the gamma function, i.e.,

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$

- CDF: no explicit form
- Mean or Expected Value: $\alpha\beta$
- Variance: $\alpha\beta^2$

Exponential Distribution

- Notation: $Y \sim \text{Exp}(\beta)$
- Parameters: $\beta > 0$ (scale)
- PDF: $f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere.} \end{cases}$
- CDF: $F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y/\beta}, & 0 \leq y < \infty \end{cases}$
- Mean or Expected Value: β
- Variance: β^2

χ^2 Distribution

- Notation: $Y \sim \chi^2(\nu)$
- Parameters: $\nu > 0$ (degrees of freedom)
- PDF: $f(y) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} y^{\nu/2-1} e^{-y/2}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere.} \end{cases}$
- CDF: no explicit form
- Mean or Expected Value: ν
- Variance: 2ν

Beta Distribution

Beta Distribution

Definition: Beta Distribution

A random variable Y is said to have a *beta probability distribution* with parameters $\alpha > 0$ and $\beta > 0$ if and only if the density function of Y is

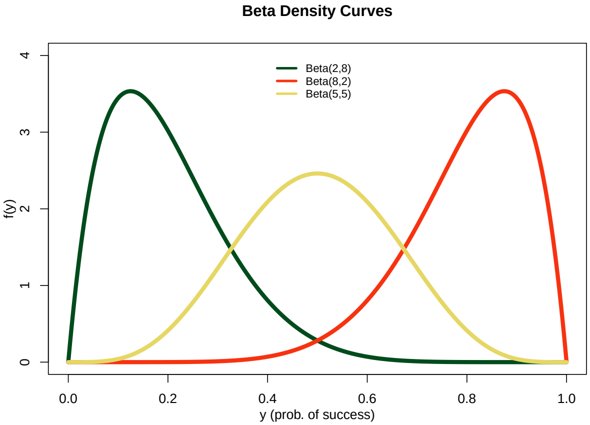
$$f(y) = \begin{cases} \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $B(\cdot)$ is the beta function, i.e.,

$$B(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

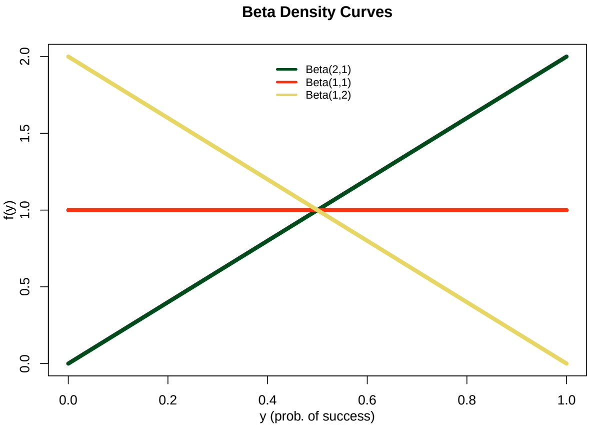
- Notation: $Y \sim \text{Beta}(\alpha, \beta)$, read as: "$Y$ is a beta random variable with parameters α and β."
- CDF: no explicit form; numerical calculations required
- Alternative PDF formulation: $f(y) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

Beta Distribution: *The Distribution for Probabilities*



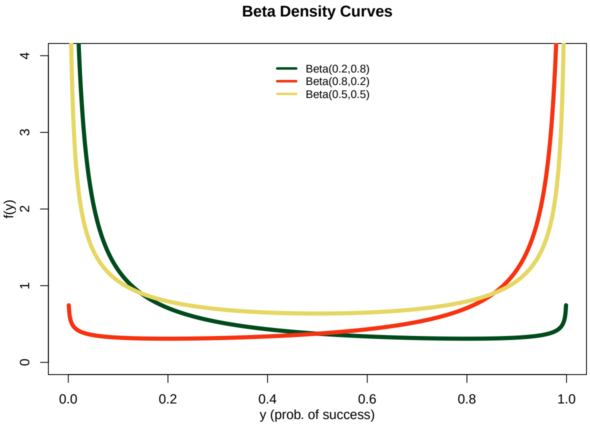
- $\alpha > 1$ & $\beta > 1$: Bell-Shaped
- If $\alpha > 1$ & $\beta > 1$ & $\alpha < \beta$, the density curve is skewed to the right. (green curve)
- If $\alpha > 1$ & $\beta > 1$ & $\alpha > \beta$, the density curve is skewed to the left. (red curve)

Beta Distribution: *The Distribution for Probabilities*



- $\alpha = 1$ & $\beta = 2$: Straight Line
- $\alpha = 2$ & $\beta = 1$: Straight Line
- When $\alpha = \beta = 1$, $\text{Beta}(1, 1) = U(0, 1)$.

Beta Distribution: *The Distribution for Probabilities*



- $\alpha < 1$ & $\beta < 1$: U-Shaped

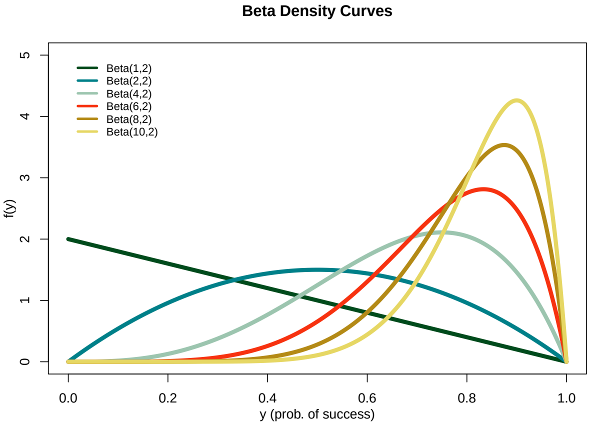
Beta-Binomial Relationship

	Random Variable	**Distribution Function**	**Probability as a...**	**Usage**
Beta	$Y \sim \text{Beta}(\alpha, \beta)$	$f(y) = \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}$	random variable	model the prob. of success
Binomial	$X \sim B(n, y)$ n: num of trials y: prob. of success	$p(x) = \binom{n}{x} y^x (1-y)^{n-x}$	fixed parameter y	model the num of successes

Matching the Beta PDF to the Binomial PMF, we can see that...

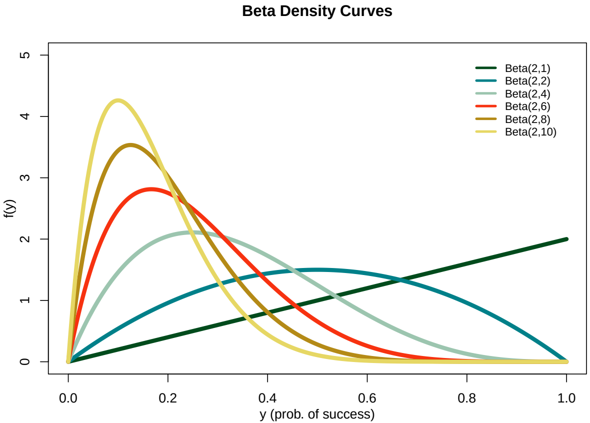
- $\alpha - 1$ can be considered as the number of successes
- $\beta - 1$ can be considered as the number of failures

Beta Distribution: The Effect of α



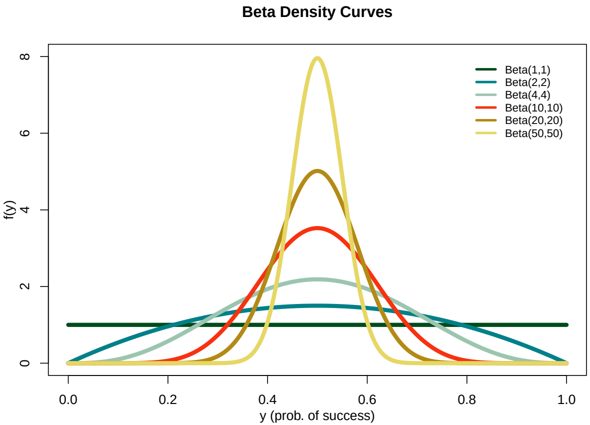
- As α increases (more successful events), the bulk of the probability distribution will shift towards the right.
  - probability of success must be closer to 1

Beta Distribution: The Effect of β



- As β increases (more failures), the bulk of the probability distribution will move towards the left.
  - probability of success must be closer to 0

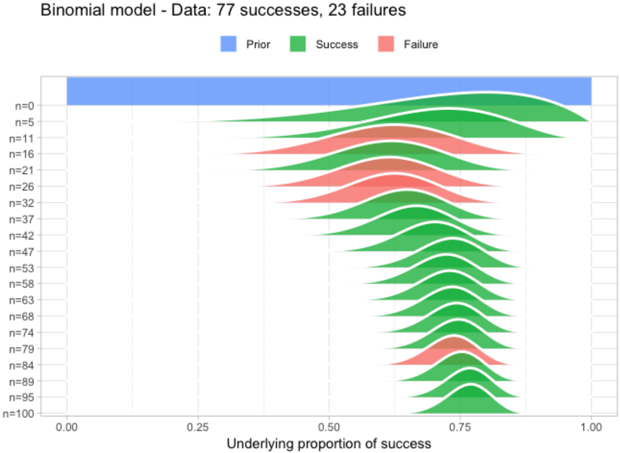
Beta Distribution: The Effect of $\alpha = \beta \to \infty$



- Increasing both α and β, narrows the probability distribution, indicating greater certainty regarding the value of the random variable.

Beta Distribution: Application

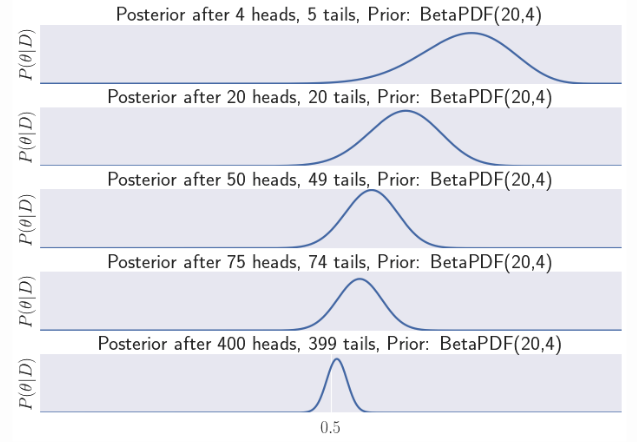
Bayesian Statistics



Source: https://www.sumsar.net/blog/2018/12/visualizing-the-beta-binomial/

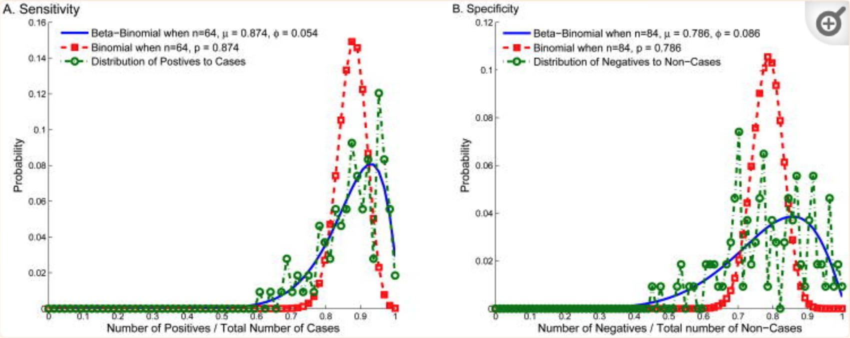
Beta Distribution: Application

Bayesian Statistics



Source: https://www.thomasjpfan.com/2015/09/bayesian-coin-flips/

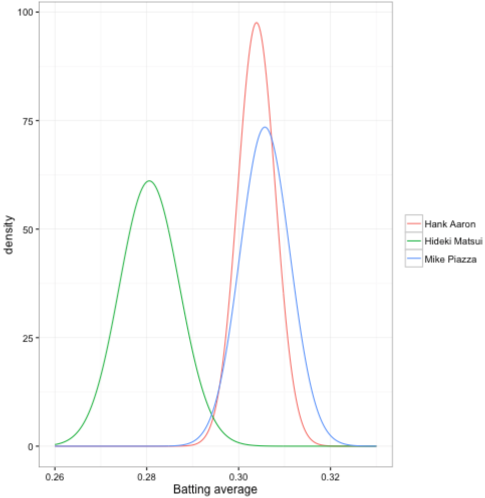
Mammography



Source: https://doi.org/10.1080/03610918.2014.960091

Beta Distribution: Application

Sports



Source: http://varianceexplained.org/r/bayesian_ab_baseball/

Beta Distribution

Theorem: Beta Distribution

If Y has a beta distribution with parameters α and β, then

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Proof:

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} y f(y) dy \\
&= \int_0^1 y \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1} dy \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 y^{\alpha}(1-y)^{\beta-1} dy \\
&= \frac{1}{B(\alpha,\beta)} B(\alpha+1,\beta) \\
&= \frac{1}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\alpha\Gamma(\alpha)}{(\alpha+\beta)\Gamma(\alpha+\beta)} \\
&= \frac{\alpha}{\alpha+\beta}.
\end{aligned}$$

def'n of expected value

$$f(y) = \begin{cases} \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 \le y \le 1, \\ 0, & \text{elsewhere}, \end{cases}$$

Beta function: $B(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy$

Beta function (alternative form): $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Gamma function recursion property: $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

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Beta Distribution

Proof:

$$
\begin{aligned}
E(Y^2) &= \int_{-\infty}^{\infty} y^2 f(y)dy && \text{def'n of expected value} \\
&= \int_0^1 y^2 \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}dy && f(y) = \begin{cases} \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases} \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 y^{\alpha+1}(1-y)^{\beta-1}dy \\
&= \frac{1}{B(\alpha,\beta)} B(\alpha+2,\beta) && \text{Beta function: } B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy \\
&= \frac{1}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} && \text{Beta function (alternative form): } B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{(\alpha+1)\Gamma(\alpha+1)}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)} && \text{Gamma function recursion property: } \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{(\alpha+1)\alpha\Gamma(\alpha)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} && \text{Gamma function recursion property: } \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \\
&= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}.
\end{aligned}
$$

$$
\begin{aligned}
V(Y) &= E(Y^2) - \{E(Y)\}^2 && \text{def'n of variance} \\
&= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2} \\
&= \frac{(\alpha+\beta)(\alpha+1)\alpha - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha^3+\alpha^2\beta+\alpha^2+\alpha\beta-\alpha^3-\alpha^2\beta-\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.
\end{aligned}
$$

□

Beta Distribution

Example 1:

Assume the proportion of time, Y, a car is broken is approximately a beta with the following density,

$$f(y) = \begin{cases} 5(1-y)^4, & 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

What are α and β?

Solution:

Matching the PDF above with the beta PDF:

$$f(y) = \begin{cases} \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

we need the following equalities:

- $\alpha - 1 = 0$. Thus, $\alpha = 1$.
- $\beta - 1 = 4$. Thus, $\beta = 5$.

Note: $B(1,5) = \frac{\Gamma(1)\Gamma(5)}{\Gamma(1+5)} = \frac{(1)4!}{5!} = \frac{1}{5}$. □

Beta Distribution

Example 2:

The maximum time to complete a certain project is 2.5 days. Suppose that the completion time as a proportion of this maximum is a Beta random variable with mean 0.4 and variance 0.2.

a. Write the PDF of the project's completion time as a proportion of the maximum time.

Solution:

- Let X be the random variable for the project's completion time as a proportion of the maximum time. $X \sim \text{Beta}(\alpha, \beta)$
  - Note: $X = \frac{\text{possible project completion time in days}}{\text{maximum project completion time in days}} \in [0, 1]$
- Given:
  - $E(X) = 0.4 \Rightarrow \frac{\alpha}{\alpha+\beta} = 0.4$ expected value formula of Beta RV
  - $V(X) = 0.2 \Rightarrow \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.2$ variance formula of Beta RV
- Solve the following system of equations:

$$\begin{cases} \frac{\alpha}{\alpha+\beta} = 0.4 \\ \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.2 \end{cases} \Rightarrow \begin{cases} \alpha = 0.4\alpha + 0.4\beta \\ \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.2 \end{cases} \Rightarrow \begin{cases} \beta = \frac{3}{2}\alpha \\ \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.2 \end{cases}$$

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Example 2:

The maximum time to complete a certain project is 2.5 days. Suppose that the completion time as a proportion of this maximum is a Beta random variable with mean 0.4 and variance 0.2.

a. Write the PDF of the project's completion time as a proportion of the maximum time.

Solution:

- Solve the following system of equations:

$$\begin{cases} \beta = \frac{3}{2}\alpha \\ \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.2 \end{cases} \quad \Rightarrow \quad \frac{\alpha\left(\frac{3}{2}\alpha\right)}{\left\{\alpha + \left(\frac{3}{2}\alpha\right)\right\}^2\left\{\alpha + \left(\frac{3}{2}\alpha\right) + 1\right\}} = 0.2$$

$$\Rightarrow \quad \frac{\frac{3}{2}\alpha^2}{\left(\frac{5}{2}\alpha\right)^2\left(\frac{5}{2}\alpha + 1\right)} = 0.2$$

$$\Rightarrow \quad \frac{6}{25}\frac{1}{\left(\frac{5}{2}\alpha + 1\right)} = \frac{1}{5}$$

$$\Rightarrow \quad \frac{6}{5} = \frac{5}{2}\alpha + 1 \Rightarrow \alpha = \frac{2}{25} \Rightarrow \beta = \frac{3}{2}\left(\frac{2}{25}\right) = \frac{3}{25}.$$

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Beta Distribution

Example 2:

The maximum time to complete a certain project is 2.5 days. Suppose that the completion time as a proportion of this maximum is a Beta random variable with mean 0.4 and variance 0.2.

a. Write the PDF of the project's completion time as a proportion of the maximum time.

Solution:

- Write the Beta PDF with $\alpha = \frac{2}{25}$ and $\beta = \frac{3}{25}$.

$$f(x) = \begin{cases} \frac{1}{B(\frac{2}{25}, \frac{3}{25})} x^{\frac{2}{25}-1}(1-x)^{\frac{3}{25}-1}, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

OR

$$f(x) = \begin{cases} \frac{\Gamma(\frac{2}{25}+\frac{3}{25})}{\Gamma(\frac{2}{25})\Gamma(\frac{3}{25})} x^{\frac{2}{25}-1}(1-x)^{\frac{3}{25}-1}, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

□

Beta Distribution

Example 2:

The maximum time to complete a certain project is 2.5 days. Suppose that the completion time as a proportion of this maximum is a Beta random variable with mean 0.4 and variance 0.2.

(b) What is the probability that the task requires more than two days to complete?

Solution:

- We are looking for $P(X > \frac{2}{2.5})$. Why? Recall: $X = \frac{\text{possible project completion time in days}}{\text{maximum project completion time in days}} \in [0, 1]$

$$\begin{aligned} P\left(X > \frac{2}{2.5}\right) &= P(X > 0.8) \\ &= \int_{0.8}^{1} f(x)dx \\ &= \int_{0.8}^{1} \frac{1}{B(\frac{2}{25}, \frac{3}{25})} x^{\frac{2}{25}-1}(1-x)^{\frac{3}{25}-1}dx. \end{aligned}$$

You can keep your answer in this form.

Beta Distribution

Example 3:

Suppose X has the beta distribution with parameters α and β, and let r and s be positive integers. Determine the value of $E\{X^r(1-X)^s\}$.

Solution:

$$E(X^r(1-X)^s) = \int_{-\infty}^{\infty} x^r(1-x)^s f(x)dx \quad \text{def'n of expected value}$$

$$= \int_0^1 x^r(1-x)^s \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$$

Beta PDF: $f(y) = \begin{cases} \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+r-1}(1-x)^{\beta+s-1}dx$$

$$= \frac{1}{B(\alpha,\beta)} B(\alpha+r, \beta+s).$$

Beta function: $B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy$ □

Student's t Distribution

Student's t Distribution

Definition: Student's t Distribution

If $Z \sim \mathcal{N}(0,1)$ and $U \sim \chi^2(\nu)$ are independent, then the random variable

$$Y = \frac{Z}{\sqrt{U/\nu}}$$

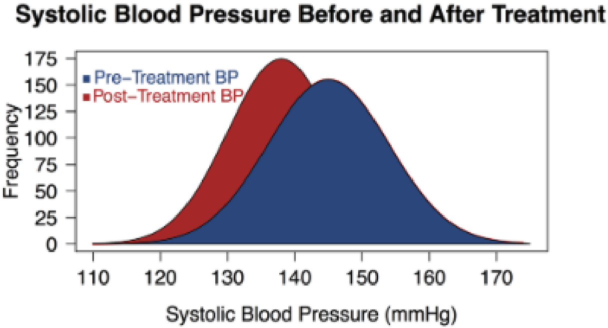
follows a *Student's t probability distribution* with ν *degrees of freedom* and its density function is

$$f(y) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty \leq y \leq \infty.$$

- Notation: $Y \sim t_{(\nu)}$, read as: "Y is a Student's t random variable with ν degrees of freedom."
- Also called t-distribution
- developed by William S. Gosset who published under the pseudonym Student

Student's t Distribution: Application

Hypothesis Testing: T-test



$$t = \frac{(x_1 - x_2)}{\sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}}}$$

Source: https://www.biologyforlife.com/t-test.html

Student's t Distribution

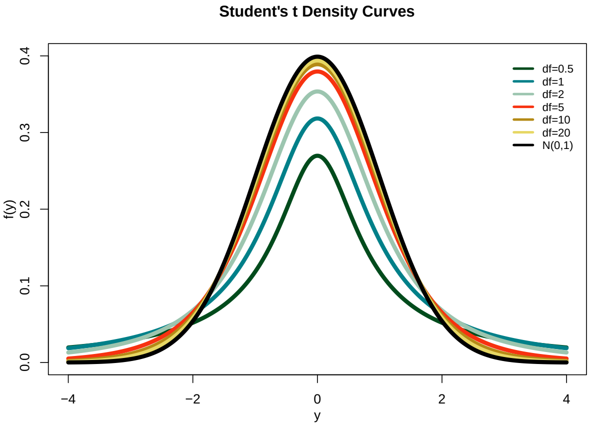
Theorem: Student's t Distribution

If Y has a Student's t distribution with ν degrees of freedom, then

$$\mu = E(Y) = 0 \quad \text{and} \quad \sigma^2 = V(Y) = \frac{\nu}{\nu - 2}.$$

Proof: Left as exercise.

Student's Distribution: The Effect of ν



- Close to the standard normal except for heavy tails.
- As ν increases, the Student's t distribution approaches the standard normal distribution.

Bonus Exercise

Questions?

Homework Exercises: 4.93, 4.95, 4.101, 4.103, 4.111

Solutions will be discussed this Friday by the TA.