STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 14: Other Expected Values

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Outline

Previously...

Special Continuous Distributions

	Uniform	Normal (Gaussian)	Standard Normal
Usage	values are equally likely	peak in the middle, then gradually tails off	peak in the middle, then gradually tails off
Parameters	$\theta_1 \in \mathbb{R}$ (minimum), $\theta_2 \in \mathbb{R}$ (maximum)	$\mu \in \mathbb{R}$ (mean), $\sigma > 0$ (standard deviation)	$\mu = 0$ (mean) $\sigma = 1$ (standard deviation)
Notation	$Y \sim U(\theta_1, \theta_2)$	$Y \sim \mathcal{N}(\mu, \sigma^2)$	$Z \sim \mathcal{N}(0, 1)$
PDF	$f(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 \leq y \leq \theta_2, \\ 0, & \text{elsewhere} \end{cases}$	$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$	$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
CDF	$F(y) = \begin{cases} 0, & y < \theta_1 \\ \frac{y - \theta_1}{\theta_2 - \theta_1}, & \theta_1 \leq y \leq \theta_2 \\ 1, & y > \theta_2. \end{cases}$	$F(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$ (no explicit form)	$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ (no explicit form)
Values of RV	$\theta_1 \leq y \leq \theta_2$	$-\infty \leq y \leq \infty$	$-\infty \leq y \leq \infty$
Mean	$\frac{\theta_1 + \theta_2}{2}$	μ	0
Variance	$\frac{(\theta_2 - \theta_1)^2}{12}$	σ^2	1

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Special Continuous Distributions

	Gamma	Exponential	χ^2
Usage	waiting time	special case of Gamma ($\alpha = 1$); memoryless	special case of Gamma ($\alpha = \frac{\nu}{2}$ and $\beta = 2$); sum of squares of std. normal hypothesis testing
Parameters	$\alpha > 0$ (shape), $\beta > 0$ (scale)	$\beta > 0$ (scale)	$\nu \in \mathbb{Z}$ (degrees of freedom)
Notation	$Y \sim \text{Gam}(\alpha, \beta)$	$Y \sim \text{Exp}(\beta)$	$Y \sim \chi^2 \nu$
PDF	$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, & 0 \leq y \leq \infty, \\ 0, & \text{elsewhere}; \end{cases}$ $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy$	$f(y) = \begin{cases} \frac{1}{\beta}e^{-y/\beta}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere} \end{cases}$	$f(y) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} y^{\nu/2-1}e^{-y/2}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere} \end{cases}$
CDF	(no explicit form)	$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y/\beta}, & 0 \leq y < \infty \end{cases}$	(no explicit form)
Values of RV	$0 \leq y \leq \infty$	$0 \leq y \leq \infty$	$0 \leq y \leq \infty$
Mean	$\alpha\beta$	β	ν
Variance	$\alpha\beta^2$	β^2	2ν

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Special Continuous Distributions

	Beta	**Student's t**
Usage	distribution for proportions and probabilities	hypothesis testing
Parameters	$\alpha > 0$ (shape), $\beta > 0$ (scale)	$\nu \in \mathbb{Z}$ (degrees of freedom)
Notation	$Y \sim \text{Beta}(\alpha, \beta)$	$Y \sim t_{(\nu)}$
PDF	$f(y) = \begin{cases} \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere;} \end{cases}$ $B(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$	$f(y) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}}$
CDF	(no explicit form)	(no explicit form)
Values of RV	$0 \leq y \leq 1$	$-\infty \leq y \leq \infty$
Mean	$\frac{\alpha}{\alpha+\beta}$	0
Variance	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$\frac{\nu}{\nu-2}$

Other Expected Values

Definition 3.12: *k*th Moment

The kth moment of a random variable Y taken about the origin is defined to be

$$E(Y^k)$$

and is denoted by μ_k'.

- The moments of the distribution are the expectations of the random variable to the integer powers.
- 1st moment, $E(Y)$: mean or expected value of a random variable
- 2nd moment, $E(Y^2)$: can be used to find the variance
  - $V(Y) = E(Y^2) - \{E(Y)\}^2$

Definition 3.13: *k*th Central Moment

The *k*th *central* moment of a random variable Y taken about its mean, is defined to be

$$E\{(Y-\mu)^k\}$$

and is denoted by μ_k.

- 1st central moment: 0
- 2nd central moment, $E\{(Y-\mu)^2\}$: variance of a random variable
- 3rd central moment, $E\{(Y-\mu)^3\}$: skewness (asymmetry)
- 4th central moment, $E\{(Y-\mu)^4\}$: kurtosis (heaviness of the tails)

Moments: Importance

- To find means and variances of random variables, and other information regarding the shape of their distributions
  - Example: What are the mean and variance of $A = \pi R^2$?

$$
\begin{aligned}
E(A) &= \pi E(R^2) \\
V(A) &= \pi^2 V(R^2) = \pi^2[E(R^4) - \{E(R^2)\}^2]
\end{aligned}
$$

    We need the 2nd, $E(R^2)$, and 4th, $E(R^4)$, moments.
  - Example: What is the skewness of Y?

$$E\{(Y-\mu)^3\} = E(Y^3 - 3\mu Y^2 + 3\mu^2 Y - \mu^3) = E(Y^3) - 3\mu E(Y^2) + 2\mu^3$$

    We need the first, μ, 2nd, $E(Y^2)$, and 3rd, $E(Y^3)$, moments.
- The moments tell you about the features of the distribution.

Moments: Importance

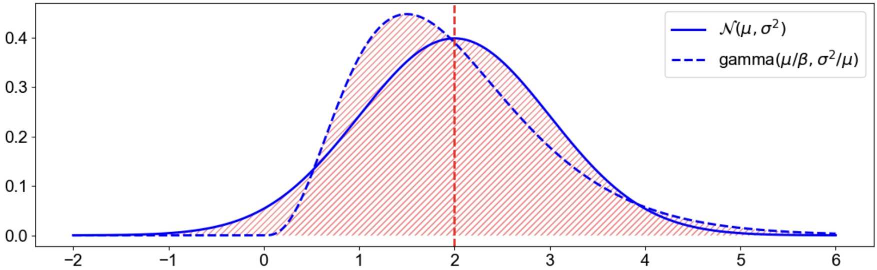

Figure 7. Two distributions with the same mean and variance but different skewnesses: a normal distribution with mean μ and variance σ^2 and a gamma distribution with parameters $\alpha = \mu/\beta$ and $\beta = \sigma^2/\mu$.

Source: https://gregorygundersen.com/blog/2020/04/11/moments/

Moments: Importance

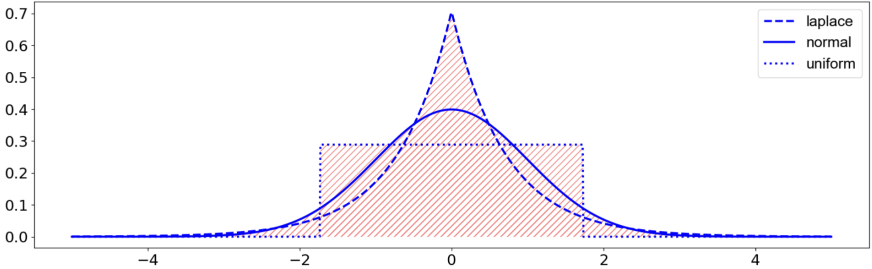

Figure 10. Laplace, normal, and uniform distributions with mean 0 and variance 1. Respectively, their excess kurtosises are 3, 0, and -1.2.

Source: https://gregorygundersen.com/blog/2020/04/11/moments/

Moment Generating Function: Why We Need MGFs

- MGFs are functions that spit out moments.
  - Faster to compute expected values using MGFs...
- Another way to identify distributions of random variables
  - PDFs and CDFs can be hard to work with
- Study convergence of distributions
  - To prove the *Central Limit Theorem*
- Special Properties of MGFs:
  - If two random variables have the same MGF, then they MUST have the same distribution.
  - The kth moment is the kth derivative of the MGF evaluated at 0.
  - The MGF for the sum of independent random variables is the product of their individual MGFs.
- Generally, MGFs are mostly used as a computational tool and has no intrinsic meaning.

Moment Generating Function

Definition: Moment Generating Function (MGF)

The *moment generating function* $m(t)$ for a random variable Y is defined to be

$$m(t) = E\left(e^{tY}\right).$$

We say that a moment-generating function for Y exists if there exists a positive constant b such that $m(t)$ is finite for $|t| \leq b$.

- Discrete Case: $m(t) = E\left(e^{tY}\right) = \sum_y e^{ty} p(y)$ def'n of expected value of discrete RV
- Continuous Case: $m(t) = E\left(e^{tY}\right) = \int_{-\infty}^{\infty} e^{ty} f(y) dy$

def'n of expected value of continuous RV

So, what's the big deal about $E\left(e^{tY}\right)$?

How do we get the moments $E(Y^k)$ from $E\left(e^{tY}\right)$?

Moment Generating Function: A Closer Look At $E\left(e^{tY}\right)$

- MGF: $m(t) = E\left(e^{tY}\right)$
- Question: How does the mean, the variance, the skewness, the kurtosis, and other higher moments appear from $E\left(e^{tY}\right)$?
  - Taylor series expansion of e^{tY}:

$$e^{tY} = 1 + \frac{tY}{1!} + \frac{t^2Y^2}{2!} + \frac{t^3Y^3}{3!} + \ldots$$

  - MGF is the expectation of e^{tY}: (alternative formulation)

$$m(t) = E\left(e^{tY}\right) = 1 + \frac{tE(Y)}{1!} + \frac{t^2E(Y^2)}{2!} + \frac{t^3E(Y^3)}{3!} + \frac{t^4E(Y^4)}{4!} + \ldots$$

- The kth moment is the COEFFICIENT of $\frac{t^k}{k!}$
  - $E(Y)$ is the coefficient of $\frac{t}{1!}$.
  - $E(Y^2)$ is the coefficient of $\frac{t^2}{2!}$.
  - $E(Y^3)$ is the coefficient of $\frac{t^3}{3!}$.
  - $E(Y^4)$ is the coefficient of $\frac{t^4}{4!}$...

Moment Generating Function: Obtaining $E(Y)$

- MGF is the expectation of the Taylor series expansion of e^{tY}: (alternative formulation)

$$m(t) = E\left(e^{tY}\right) = 1 + \frac{tE(Y)}{1!} + \frac{t^2E(Y^2)}{2!} + \frac{t^3E(Y^3)}{3!} + \frac{t^4E(Y^4)}{4!} + \dots$$

- mean $E(Y) \Leftrightarrow$ 1st moment $\Leftrightarrow$ 1st derivative of $m(t)$ at $t = 0$
  - Taking the 1st derivative of $m(t)$ with respect to t:

$$\frac{dm(t)}{dt} = m'(t) = E(Y) + \frac{2tE(Y^2)}{2!} + \frac{3t^2E(Y^3)}{3!} + \frac{4t^3E(Y^4)}{4!} + \dots$$

  - Setting $t = 0$:

$$\begin{aligned}\left.\frac{dm(t)}{dt}\right|_{t=0} = m'(0) &= E(Y) + \frac{2(0)E(Y^2)}{2!} + \frac{3(0)^2E(Y^3)}{3!} + \frac{4(0)^3E(Y^4)}{4!} + \dots \\ &= E(Y).\end{aligned}$$

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Moment Generating Function: Obtaining $E(Y^2)$

- MGF is the expectation of the Taylor series expansion of e^{tY}: (alternative formulation)

$$m(t) = E\left(e^{tY}\right) = 1 + \frac{tE(Y)}{1!} + \frac{t^2E(Y^2)}{2!} + \frac{t^3E(Y^3)}{3!} + \frac{t^4E(Y^4)}{4!} + \dots$$

- $E(Y^2) \Leftrightarrow$ 2nd moment $\Leftrightarrow$ 2nd derivative of $m(t)$ at $t = 0$
  - Taking the 2nd derivative of $m(t)$ with respect to t:

$$\begin{aligned} \frac{d}{dt}\left\{\frac{dm(t)}{dt}\right\} &= \frac{d}{dt}\left\{m'(t)\right\} = \frac{d}{dt}\left\{E(Y) + \frac{2tE(Y^2)}{2!} + \frac{3t^2E(Y^3)}{3!} + \frac{4t^3E(Y^4)}{4!} + \dots\right\} \\ \frac{d^2m(t)}{dt^2} &= m''(t) = E(Y^2) + \frac{3(2)tE(Y^3)}{3!} + \frac{4(3)t^2E(Y^4)}{4!} + \dots \end{aligned}$$

  - Setting $t = 0$:

$$\begin{aligned} \left.\frac{d^2m(t)}{dt^2}\right|_{t=0} = m''(0) &= E(Y^2) + \frac{3(2)(0)E(Y^3)}{3!} + \frac{4(3)(0)^2E(Y^4)}{4!} + \dots \\ &= E(Y^2). \end{aligned}$$

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Moment Generating Function: Obtaining $E(Y^3)$

- MGF is the expectation of the Taylor series expansion of e^{tY}: (alternative formulation)

$$m(t) = E\left(e^{tY}\right) = 1 + \frac{tE(Y)}{1!} + \frac{t^2E(Y^2)}{2!} + \frac{t^3E(Y^3)}{3!} + \frac{t^4E(Y^4)}{4!} + \dots$$

- $E(Y^3) \Leftrightarrow$ 3rd moment $\Leftrightarrow$ 3rd derivative of $m(t)$ at $t = 0$
  - Taking the 3rd derivative of $m(t)$ with respect to t:

$$\begin{aligned} \frac{d}{dt}\left\{\frac{d^2m(t)}{dt^2}\right\} &= \frac{d}{dt}\left\{m''(t)\right\} = \frac{d}{dt}\left\{E(Y^2) + \frac{3(2)tE(Y^3)}{3!} + \frac{4(3)t^2E(Y^4)}{4!} + \dots\right\} \\ \frac{d^3m(t)}{dt^3} &= m^{(3)}(t) = E(Y^3) + \frac{4(3)(2)tE(Y^4)}{4!} + \dots \end{aligned}$$

  - Setting $t = 0$:

$$\begin{aligned} \left.\frac{d^3m(t)}{dt^3}\right|_{t=0} = m^{(3)}(0) &= E(Y^3) + \frac{4(3)(2)(0)E(Y^4)}{4!} + \dots \\ &= E(Y^3). \end{aligned}$$

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Moment Generating Function: Obtaining $E(Y^4)$

- MGF is the expectation of the Taylor series expansion of e^{tY}: (alternative formulation)

$$m(t) = E\left(e^{tY}\right) = 1 + \frac{tE(Y)}{1!} + \frac{t^2E(Y^2)}{2!} + \frac{t^3E(Y^3)}{3!} + \frac{t^4E(Y^4)}{4!} + \ldots$$

- $E(Y^4) \Leftrightarrow$ 4th moment $\Leftrightarrow$ 4th derivative of $m(t)$ at $t = 0$
  - Taking the 4th derivative of $m(t)$ with respect to t:

$$\begin{aligned} \frac{d}{dt}\left\{\frac{d^3m(t)}{dt^3}\right\} &= \frac{d}{dt}\left\{m^{(3)}(t)\right\} = \frac{d}{dt}\left\{E(Y^3) + \frac{4(3)(2)tE(Y^4)}{4!} + \ldots\right\} \\ \frac{d^4m(t)}{dt^4} &= m^{(4)}(t) = E(Y^4) + \ldots \end{aligned}$$

  - Setting $t = 0$:

$$\left.\frac{d^4m(t)}{dt^4}\right|_{t=0} = m^{(4)}(0) \quad = \quad E(Y^4).$$

Moment Generating Function

Theorem 3.12

If $m(t)$ exists, then for any positive integer k,

$$\left.\frac{d^k m(t)}{dt^k}\right|_{t=0} = m_Y^{(k)}(0) = E(Y^k) = \mu_k'.$$

In other words, the kth moment of a random variable is the kth derivative of its moment generating function with respect to t and evaluated at 0.

Moment Generating Function: Importance of MGFs

- WAIT... but we can calculate moments using the definition of expected value...
  - Discrete Case: $E(Y^k) = \sum_y y^k p(y)$
  - Continuous Case: $E(Y^k) = \int_{-\infty}^{\infty} y^k f(y)dy$
- Why do we need MGFs exactly? For easier computations
- Example: Exponential Distribution PDF: $f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$
  - Deriving the MGF:

$$
\begin{aligned}
m(t) = E(e^{tY}) &= \int_{-\infty}^{\infty} e^{ty} f(y)dy = \int_0^{\infty} e^{ty} \frac{1}{\beta} e^{-y/\beta} dy \\
&= \frac{1}{\beta} \int_0^{\infty} e^{-y/\beta + ty} dy = \frac{1}{\beta} \int_0^{\infty} e^{-y\left(\frac{1}{\beta} - t\right)} dy \quad \text{Note: *} \\
&= \frac{1}{\beta} \left\{ -\frac{1}{\left(\frac{1}{\beta} - t\right)} e^{-y\left(\frac{1}{\beta} - t\right)} \right\} \Bigg|_0^{\infty} = \frac{1}{\beta} \left\{ \frac{1}{\left(\frac{1}{\beta} - t\right)} \right\} = \frac{1}{1 - \beta t}.
\end{aligned}
$$

*$1/\beta - t$ must be positive for this integral to converge $\Rightarrow 1/\beta - t > 0 \Rightarrow t < 1/\beta$.

Moment Generating Function: Importance of MGFs

- Why do we need MGFs exactly? For easier computations
- Example: Exponential Distribution PDF: $f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$
  - MGF: $m(t) = \frac{1}{1-\beta t}$

$$m'(t) = \frac{d}{dt}\left(\frac{1}{1-\beta t}\right) = \frac{\beta}{(1-\beta t)^2}$$

$$m''(t) = \frac{d}{dt}\{m'(t)\} = \frac{d}{dt}\left\{\frac{\beta}{(1-\beta t)^2}\right\} = \frac{(\beta)\{2(1-\beta t)(\beta)\}}{(1-\beta t)^4} = \frac{2(\beta^2-\beta^3 t)}{(1-\beta t)^4}$$

$$m^{(3)}(t) = \frac{d}{dt}\{m''(t)\} = \frac{d}{dt}\left\{\frac{2(\beta^2-\beta^3 t)}{(1-\beta t)^4}\right\}$$

$$= \frac{(1-\beta t)^4(-2\beta^3) - 2(\beta^2-\beta^3 t)(-4\beta)(1-\beta t)^3}{(1-\beta t)^8}$$

Which one is easier to compute?

	Using MGF (derivatives)	Using PDF (integrals)
$E(Y)$	$m'(0) = \beta$	$\int_0^\infty y \frac{1}{\beta} e^{-y/\beta} dy$
$E(Y^2)$	$m''(0) = 2\beta^2$	$\int_0^\infty y^2 \frac{1}{\beta} e^{-y/\beta} dy$
$E(Y^3)$	$m^{(3)}(0) = 6\beta^3$	$\int_0^\infty y^3 \frac{1}{\beta} e^{-y/\beta} dy$

Moment Generating Function

Example 1:

A random variable X has the MGF $m(t) = \frac{1}{1-t}$, defined for any $t < 1$. What is $P(X < 1)$?

Solution:

- In Slides 21 & 22, we know that the MGF of an exponential RV is $m(t) = \frac{1}{1-\beta t}$.
- Matching the MGF of an exponential RV to the MGF above, we realize that X is an exponential RV with $\beta = 1$.
- Computing $P(X < 1)$:

$$\begin{aligned} P(X<1) &= \int_{-\infty}^{1} f(x)dx \quad \text{probability = area under the PDF} \\ &= \int_{0}^{1} e^{-x}dx \quad \text{exponential PDF: } f(y) = \frac{1}{\beta}e^{-y/\beta} \text{ for } y \geq 0 \\ &= (-e^{-x})\Big|_0^1 = 1 - e^{-1}. \end{aligned}$$

Moment Generating Function

Example 2:

Consider a random variable X has PMF

$$f(x) = \frac{6}{3^x}, \quad x = 2, 3, 4, \ldots$$

(a) Find the MGF of X.

Solution:

$$\begin{aligned} m(t) = E(e^{tX}) &= \sum_x e^{tx} f(x) = \sum_{x=2}^{\infty} e^{tx} \frac{6}{3^x} = 6 \sum_{x=2}^{\infty} \left(\frac{e^t}{3}\right)^x \\ &= 6 \left\{ \frac{\left(\frac{e^t}{3}\right)^2}{1 - \frac{e^t}{3}} \right\} \quad \text{sum of geometric series with common ratio } 0 \le \frac{e^t}{3} < 1. \\ &= 6 \left(\frac{\frac{e^{2t}}{9}}{\frac{3-e^t}{3}} \right) = 6 \left(\frac{e^{2t}}{9} \right) \left(\frac{3}{3 - e^t} \right) = \frac{2e^{2t}}{3 - e^t}, \quad t < \log(3). \end{aligned}$$

The restriction $t < \log(3)$ is required in order for the infinite geometric sum to converge. Remember common ratio must be $0 \le \frac{e^t}{3} < 1 \Rightarrow 0 \le e^t < 3 \Rightarrow -\infty \le t < \log(3)$. □

Example 2:

Consider a random variable X has PMF

$$f(x) = \frac{6}{3^x}, \quad x = 2, 3, 4, \ldots$$

(b) Find $E(X)$.

Solution:

$$m(t) = \frac{2e^{2t}}{3 - e^t} \quad \text{From Part a)}$$

$$m'(t) = \frac{(3 - e^t)4e^{2t} - 2e^{2t}(-e^t)}{(3 - e^t)^2}$$

$$E(X) = m'(0) = \frac{(3 - e^{(0)})4e^{2(0)} - 2e^{2(0)}(-e^{(0)})}{(3 - e^{(0)})^2}$$

Mean is the first derivative of MGF evaluated at $t = 0$

$$= \frac{(3-1)4 - 2(-1)}{(3-1)^2} = \frac{(2)4 + 2}{2^2} = \frac{10}{4} = \frac{5}{2}.$$

□

Moment Generating Function

Example 3:

The random variable Y has MGF

$$m(t) = 0.1e^{3t} + 0.3e^{2t} + 0.5e^{t} + 0.1.$$

a. Find the PMF of Y.

Solution:

Matching the MGF formula to the MGF above:

$$m(t) = E\left(e^{tY}\right) = \sum_y e^{ty} p(y)$$

$$m(t) = 0.1e^{3t} + 0.3e^{2t} + 0.5e^{1t} + 0.1e^{0t}$$

Therefore, $p(y) = \begin{cases} 0.1, & \text{if y} = 0, \\ 0.5, & \text{if y} = 1, \\ 0.3, & \text{if y} = 2, \\ 0.1, & \text{if y} = 3. \end{cases}$

Example 3:

The random variable Y has MGF

$$m(t) = 0.1e^{3t} + 0.3e^{2t} + 0.5e^{t} + 0.1.$$

(b) Find $E(Y)$.

Solution:

$$\begin{aligned} m'(t) &= 0.1(3)e^{3t} + 0.3(2)e^{2t} + 0.5e^{t} \\ E(Y) = m'(0) &= 0.3e^{3(0)} + 0.6e^{2(0)} + 0.5e^{0} \end{aligned}$$

Mean is the first derivative of MGF evaluated at $t = 0$

$$= 1.4.$$

Moment Generating Function

Example 3:

The random variable Y has MGF

$$m(t) = 0.1e^{3t} + 0.3e^{2t} + 0.5e^{t} + 0.1.$$

- (c) Find $V(Y)$.

Solution:

$$\begin{aligned}
m'(t) &= 0.1(3)e^{3t} + 0.3(2)e^{2t} + 0.5e^{t} \\
m''(t) &= 0.1(3)(3)e^{3t} + 0.3(2)(2)e^{2t} + 0.5e^{t} \\
E(Y^2) = m''(0) &= 0.9e^{3(0)} + 1.2e^{2(0)} + 0.5e^{0}
\end{aligned}$$

$E(Y^2)$ is the 2nd derivative of MGF evaluated at $t = 0$

$$\begin{aligned}
&= 2.6. \\
V(Y) &= E(Y^2) - \{E(Y)\}^2 \quad \text{def'n of variance} \\
&= 2.6 - 1.4^2 = 0.64.
\end{aligned}$$

MGFs of Discrete Distributions

MGF: Bernoulli

- PMF: $p(y) = p^y(1-p)^{1-y}$, $y = 0, 1$.
- MGF:
$$\begin{aligned} m(t) = E\left(e^{tY}\right) &= \sum_y e^{ty} p(y) \\ &= e^{t(0)} p(0) + e^{t(1)} p(1) \\ &= (1-p) + pe^t. \end{aligned}$$
- Mean or Expected Value:
$$\begin{aligned} m'(t) &= pe^t. \\ E(Y) &= m'(0) = p. \end{aligned}$$
- Variance:
$$\begin{aligned} m''(t) &= pe^t. \\ E(Y^2) &= m''(0) = p. \\ V(Y) &= E(Y^2) - \{E(Y)\}^2 = p - p^2 = p(1-p). \end{aligned}$$

MGF: Binomial

- PMF: $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$, $y = 0, 1, \ldots, n$.
- MGF:

$$
\begin{aligned}
m(t) = E\left(e^{tY}\right) &= \sum_y e^{ty} p(y) = \sum_{y=0}^{n} e^{ty} \binom{n}{y} p^y (1-p)^{n-y} \\
&= \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y (1-p)^{n-y} \\
&= (pe^t + 1 - p)^n.
\end{aligned}
$$

Binomial Theorem: $(a+b)^n = \sum_{y=0}^{n} \binom{n}{y} a^y b^{n-y}$

- Mean or Expected Value:

$$
\begin{aligned}
m'(t) &= n(pe^t + 1 - p)^{n-1}(pe^t). \\
E(Y) &= m'(0) = np.
\end{aligned}
$$

- Variance:

$$
\begin{aligned}
m''(t) &= n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1}(pe^t). \\
E(Y^2) &= m''(0) = n(n-1)p^2 + np. \\
V(Y) &= E(Y^2) - \{E(Y)\}^2 = n(n-1)p^2 + np - (np)^2 = np(1-p).
\end{aligned}
$$

MGF: Geometric

- PMF: $p(y) = (1-p)^{y-1}p$, $y = 1, 2, \ldots$.
- MGF: $m(t) = E\left(e^{tY}\right) = \sum_y e^{ty}p(y) = \sum_{y=1}^{\infty} e^{ty}(1-p)^{y-1}p$

$$= pe^t \sum_{y=1}^{\infty} e^{t(y-1)}(1-p)^{y-1} = pe^t \sum_{y=1}^{\infty}\{e^t(1-p)\}^{y-1}$$

$$= pe^t \frac{1}{1-e^t(1-p)}.$$ sum of geometric series with ratio $0 \le e^t(1-p) < 1$

- Mean or Expected Value:

$$m'(t) = \frac{\{1-e^t(1-p)\}(pe^t) - pe^t\{-e^t(1-p)\}}{\{1-e^t(1-p)\}^2} = \frac{pe^t}{\{1-e^t(1-p)\}^2}.$$

$$E(Y) = m'(0) = \frac{p}{p^2} = \frac{1}{p}.$$

- Variance:

$$m''(t) = \frac{\{1-e^t(1-p)\}^2(pe^t) - (pe^t)2\{1-e^t(1-p)\}\{-e^t(1-p)\}}{\{1-e^t(1-p)\}^4}.$$

$$E(Y^2) = m''(0) = \frac{p^3 - 2p^3 + 2p^2}{p^4} = \frac{-p^3 + 2p^2}{p^4}.$$

$$V(Y) = E(Y^2) - \{E(Y)\}^2 = \frac{-p^3+2p^2}{p^4} - \frac{1}{p^2} = \frac{-p^3+2p^2-p^2}{p^4}$$

$$= \frac{p^2(1-p)}{p^4} = \frac{1-p}{p^2}.$$

MGF: Poisson

- PMF: $p(y) = \frac{\lambda^y}{y!} e^{-\lambda}$, $y = 0, 1, 2, \ldots$ and $\lambda > 0$.
- MGF:

$$\begin{aligned} m(t) = E\left(e^{tY}\right) &= \sum_y e^{ty} p(y) = \sum_{y=0}^{\infty} e^{ty} \frac{\lambda^y}{y!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{y=0}^{\infty} \frac{(e^t \lambda)^y}{y!} = e^{-\lambda} e^{e^t \lambda}. \\ &= e^{\lambda(e^t - 1)}. \end{aligned}$$

Taylor series expansion of $e^{e^t \lambda}$

- Mean or Expected Value:

$$\begin{aligned} m'(t) &= e^{\lambda(e^t-1)} \lambda e^t. \\ E(Y) &= m'(0) = \lambda. \end{aligned}$$

- Variance:

$$\begin{aligned} m''(t) &= e^{\lambda(e^t-1)} (\lambda e^t)^2 + e^{\lambda(e^t-1)} \lambda e^t. \\ E(Y^2) &= m''(0) = \lambda^2 + \lambda. \\ V(Y) &= E(Y^2) - \{E(Y)\}^2 = \lambda^2 + \lambda - \lambda^2 = \lambda. \end{aligned}$$

MGFs of Continuous Distributions

MGF: Uniform

- PDF: $f(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2, \\ 0, & \text{elsewhere} \end{cases}$
- MGF:

$$\begin{aligned} m(t) = E\left(e^{tY}\right) &= \int_{-\infty}^{\infty} e^{ty} f(y) dy = \int_{\theta_1}^{\theta_2} e^{ty} \frac{1}{\theta_2 - \theta_1} dy \\ &= \frac{1}{\theta_2 - \theta_1} \frac{e^{ty}}{t}\Big|_{\theta_1}^{\theta_2} = \frac{1}{\theta_2 - \theta_1} \frac{e^{t\theta_2} - e^{t\theta_1}}{t} \\ &= \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}, \quad t \neq 0. \end{aligned}$$

If $t = 0$, $\lim_{t \to 0} \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)} = \lim_{t \to 0} \frac{0}{0}$. (indeterminate form, Apply *L'Hospital's Rule*)

$\Rightarrow \lim_{t \to 0} \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)} = \lim_{t \to 0} \frac{\theta_2 e^{t\theta_2} - \theta_1 e^{t\theta_1}}{\theta_2 - \theta_1} = 1.$

Therefore, the MGF of $Y \sim U(\theta_1, \theta_2)$ is

$$m(t) = \begin{cases} \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}, & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

MGF: Uniform (cont'd)

- MGF: $$m(t) = \begin{cases} \frac{e^{t\theta_2}-e^{t\theta_1}}{t(\theta_2-\theta_1)}, & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

- Mean or Expected Value:

$$m'(t) = \frac{t(\theta_2-\theta_1)(\theta_2 e^{t\theta_2} - \theta_1 e^{t\theta_1}) - (e^{t\theta_2} - e^{t\theta_1})(\theta_2-\theta_1)}{\{t(\theta_2-\theta_1)\}^2}$$

$$= \frac{t(\theta_2 e^{t\theta_2} - \theta_1 e^{t\theta_1}) - (e^{t\theta_2} - e^{t\theta_1})}{t^2(\theta_2-\theta_1)}$$

$$E(Y) = \lim_{t\to 0} m'(t) = \frac{0}{0} \quad \text{(indeterminate form, Apply } \textit{L'Hospital's Rule}\text{)}$$

$$\Rightarrow E(Y) = \lim_{t\to 0} \frac{(\theta_2 e^{t\theta_2}-\theta_1 e^{t\theta_1}) + t(\theta_2^2 e^{t\theta_2} - \theta_1^2 e^{t\theta_1}) - (\theta_2 e^{t\theta_2}-\theta_1 e^{t\theta_1})}{2t(\theta_2-\theta_1)}$$

$$= \lim_{t\to 0} \frac{t(\theta_2^2 e^{t\theta_2} - \theta_1^2 e^{t\theta_1})}{2t(\theta_2-\theta_1)} = \lim_{t\to 0} \frac{\theta_2^2 e^{t\theta_2} - \theta_1^2 e^{t\theta_1}}{2(\theta_2-\theta_1)} = \frac{\theta_2^2-\theta_1^2}{2(\theta_2-\theta_1)}$$

$$= \frac{(\theta_2-\theta_1)(\theta_2+\theta_1)}{2(\theta_2-\theta_1)} = \frac{\theta_2+\theta_1}{2}.$$

MGF: Uniform (cont'd)

- MGF:
$$m(t) = \begin{cases} \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}, & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$
- $m'(t) = \frac{t(\theta_2 e^{t\theta_2} - \theta_1 e^{t\theta_1}) - (e^{t\theta_2} - e^{t\theta_1})}{t^2(\theta_2 - \theta_1)}$ (derived in the previous slide)
- It is quite lengthy to keep computing the derivatives!
- Alt. approach: remember that the moments are the coefficients of $\frac{t^k}{k!}$

$$m(t) = 1 + \frac{tE(Y)}{1!} + \frac{t^2 E(Y^2)}{2!} + \frac{t^3 E(Y^3)}{3!} + \dots$$ Recall from Slide 15 the alternative formulation of MGF

$$m(t) = \frac{1}{t(\theta_2 - \theta_1)}(e^{t\theta_2} - e^{t\theta_1})$$

$$= \frac{1}{t(\theta_2 - \theta_1)}\Big\{1 + \frac{\theta_2 t}{1!} + \frac{\theta_2^2 t^2}{2!} + \frac{\theta_2^3 t^3}{3!} + \dots$$ Taylor's expansion of $e^{t\theta_2}$

$$- \Big(1 + \frac{\theta_1 t}{1!} + \frac{\theta_1^2 t^2}{2!} + \frac{\theta_1^3 t^3}{3!} + \dots\Big)\Big\}$$ Taylor's expansion of $e^{t\theta_1}$

$$= \frac{1}{t(\theta_2 - \theta_1)}\Big\{(\theta_2 - \theta_1)\frac{t}{1!} + (\theta_2^2 - \theta_1^2)\frac{t^2}{2!} + (\theta_2^3 - \theta_1^3)\frac{t^3}{3!} + \dots\Big\}$$

$$= 1 + \frac{(\theta_2^2 - \theta_1^2)}{\theta_2 - \theta_1}\frac{t}{2!} + \frac{(\theta_2^3 - \theta_1^3)}{(\theta_2 - \theta_1)}\frac{t^2}{3!} + \dots$$

- Therefore, $E(Y^k) = \frac{1}{k+1}\frac{\theta_2^{k+1} - \theta_1^{k+1}}{\theta_2 - \theta_1}$.

MGF: Uniform (cont'd)

- MGF:

$$m(t) = \begin{cases} \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}, & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

- $E(Y^k) = \frac{1}{k+1} \frac{\theta_2^{k+1} - \theta_1^{k+1}}{\theta_2 - \theta_1}$ (derived in the previous slide)
- Variance:

$$E(Y^2) = \frac{1}{3}\frac{(\theta_2^3 - \theta_1^3)}{(\theta_2 - \theta_1)} = \frac{(\theta_2 - \theta_1)(\theta_2^2 + \theta_2\theta_1 + \theta_1^2)}{3(\theta_2 - \theta_1)}$$ diff. of two cubes

$$= \frac{\theta_2^2 + \theta_2\theta_1 + \theta_1^2}{3}$$

$$V(Y) = E(Y^2) - \{E(Y)\}^2$$

$$= \frac{\theta_2^2 + \theta_2\theta_1 + \theta_1^2}{3} - \left(\frac{\theta_2 + \theta_1}{2}\right)^2$$

$$= \frac{(\theta_2 - \theta_1)^2}{12}.$$

See Lec 10, Slide 14 for complete derivation.

MGF: Standard Normal

- PDF: $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty \leq z \leq \infty$
- MGF:

$$
\begin{aligned}
m(t) = E\left(e^{tZ}\right) &= \int_{-\infty}^{\infty} e^{tz}\phi(z)dz = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2-2tz)}dz \\
&= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2-2tz)}dz \quad \text{multiply a factor of 1} \\
&= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2-2tz+t^2)}dz \\
&= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2}dz \quad \text{This is the } \mathcal{N}(\mu = t, \sigma^2 = 1) \text{ PDF} \\
&= e^{\frac{t^2}{2}}(1) \quad \text{Gaussian PDF integrates to 1} \\
&= e^{\frac{t^2}{2}}.
\end{aligned}
$$

MGF: Standard Normal (cont'd)

- PDF: $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty \leq z \leq \infty$
- MGF: $m(t) = e^{\frac{t^2}{2}}$
- Mean or Expected Value:
$$\begin{aligned} m'(t) &= te^{\frac{t^2}{2}}. \\ E(Z) &= m'(0) = 0. \end{aligned}$$
- Variance:
$$\begin{aligned} m''(t) &= e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}} \\ E(Z^2) &= m''(0) = 1. \\ V(Z) &= E(Z^2) - \{E(Z)\}^2 = 1 + 0^2 = 1. \end{aligned}$$

MGF: Normal (Gaussian) – Preliminaries

Theorem: MGF of a Linear Transformation

For any constants a and b, the MGF of the random variable $aX + b$ is

$$m_{aX+b}(t) = e^{bt} m_X(at)$$

Proof:

$$\begin{aligned}
m_{aX+b}(t) &= E\left\{e^{t(aX+b)}\right\} && \text{def'n of MGF} \\
&= E\left(e^{taX} e^{tb}\right) \\
&= e^{tb} E\left(e^{taX}\right) && \text{expected value of a scaled random variable} \\
&= e^{tb} m_X(at). && \text{def'n of MGF}
\end{aligned}$$

MGF: Normal (Gaussian)

- Suppose $Y \sim \mathcal{N}(\mu, \sigma^2)$.
- Since $Y = \sigma Z + \mu$, where Z is a standard normal RV, by the preceeding theorem, its MGF is given by:

$$
\begin{aligned}
m_Y(t) &= m_{\sigma Z+\mu}(t) && \text{Standard normal linear transformation} \\
&= e^{\mu t} m_Z(\sigma t) && \text{Theorem: } m_{aX+b}(t) = e^{bt} m_X(at) \\
&= e^{\mu t} e^{\frac{(\sigma t)^2}{2}} && \text{MGF of standard normal: } m(t) = e^{\frac{t^2}{2}} \\
&= e^{\mu t + \frac{\sigma^2 t^2}{2}}.
\end{aligned}
$$

MGF: Normal (Gaussian)

- Note that we have not yet proved that the mean of a Gaussian random variable is indeed μ and that the variance is σ^2.
- $E(Y) = \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$ (direct integration is lengthy!)
- $E(Y^2) = \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$ (direct integration is lengthy!)
- MGF: $e^{\mu t + \frac{\sigma^2 t^2}{2}}$ (derived in the previous slide)
- Mean or Expected Value:

$$\begin{aligned} m'(t) &= \mu e^{\mu t + \frac{\sigma^2 t^2}{2}} + \sigma^2 t e^{\mu t + \frac{\sigma^2 t^2}{2}} = \left(\mu + \sigma^2 t\right) e^{\mu t + \frac{\sigma^2 t^2}{2}}. \\ E(Y) &= m'(0) = \mu. \end{aligned}$$

- Variance:

$$\begin{aligned} m''(t) &= \sigma^2 e^{\mu t + \frac{\sigma^2 t^2}{2}} + \left(\mu + \sigma^2 t\right)^2 e^{\mu t + \frac{\sigma^2 t^2}{2}} \\ &= \left\{\sigma^2 + \left(\mu + \sigma^2 t\right)^2\right\} e^{\mu t + \frac{\sigma^2 t^2}{2}}. \\ E(Y^2) &= m''(0) = \sigma^2 + \mu^2. \\ V(Y) &= E(Y^2) - \{E(Y)\}^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2. \end{aligned}$$

MGF: Exponential

- PDF: $f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$
- MGF: $m(t) = \frac{1}{1-\beta t}, \quad t < \frac{1}{\beta},$ (Derived in Slide 21)
- Mean or Expected Value:

$$\begin{aligned} m'(t) &= \frac{\beta}{(1-\beta t)^2}. \\ E(Y) &= m'(0) = \beta. \end{aligned}$$

- Variance:

$$\begin{aligned} m''(t) &= \frac{d}{dt}\{m'(t)\} = \frac{d}{dt}\left\{\frac{\beta}{(1-\beta t)^2}\right\} \\ &= \frac{(\beta)\{2(1-\beta t)(\beta)\}}{(1-\beta t)^4} = \frac{2(\beta^2 - \beta^3 t)}{(1-\beta t)^4} \\ E(Y^2) &= m''(0) = 2\beta^2. \\ V(Y) &= E(Y^2) - \{E(Y)\}^2 = 2\beta^2 - \beta^2 = \beta^2. \end{aligned}$$

MGF: Gamma

- PDF: $f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}, & 0 \le y \le \infty, \\ 0, & \text{elsewhere}; \end{cases}$
- MGF:

$$\begin{aligned} m(t) &= E\left(e^{tY}\right) = \int_{-\infty}^{\infty} e^{ty} f(y)dy = \int_0^\infty e^{ty} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)} dy \\ &= \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty e^{-y\left(\frac{1}{\beta}-t\right)} y^{\alpha-1} dy \\ &= \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty e^{-y\left(\frac{1-\beta t}{\beta}\right)} y^{\alpha-1} dy = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty e^{-y/\left(\frac{\beta}{1-\beta t}\right)} y^{\alpha-1} dy \\ &= \frac{1}{\beta^\alpha\Gamma(\alpha)} \left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha) \int_0^\infty \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha)} e^{-y/\left(\frac{\beta}{1-\beta t}\right)} y^{\alpha-1} dy \end{aligned}$$

multiply a factor of 1

$$\begin{aligned} &= \frac{1}{\beta^\alpha\Gamma(\alpha)} \left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha) \int_0^\infty \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha)} e^{-y/\left(\frac{\beta}{1-\beta t}\right)} y^{\alpha-1} dy \\ &= \frac{1}{(1-\beta t)^\alpha}(1), \quad \text{PDF of Gam}\left(\alpha, \frac{\beta}{1-\beta t}\right) \text{ integrates to 1} \end{aligned}$$

provided $1 - \beta t > 0 \Rightarrow t < \frac{1}{\beta}$.

MGF: Gamma (cont'd)

- PDF: $f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, & 0 \le y \le \infty, \\ 0, & \text{elsewhere;} \end{cases}$
- MGF: $m(t) = \frac{1}{(1-\beta t)^\alpha}, \quad t < \frac{1}{\beta}$
- Mean or Expected Value:

$$\begin{aligned} m'(t) &= \frac{d}{dt}\{(1-\beta t)^{-\alpha}\} = -\alpha(1-\beta t)^{-\alpha-1}(-\beta). \\ E(Y) &= m'(0) = \alpha\beta. \end{aligned}$$

- Variance:

$$\begin{aligned} m''(t) &= \frac{d}{dt}\{m'(t)\} = \frac{d}{dt}\left\{-\alpha(1-\beta t)^{-\alpha-1}(-\beta)\right\} \\ &= -\alpha(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)(-\beta) \\ E(Y^2) &= m''(0) = \alpha(1+\alpha)\beta^2. \\ V(Y) &= E(Y^2) - \{E(Y)\}^2 = \alpha(1+\alpha)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2. \end{aligned}$$

Questions?

Homework Exercises: 4.139, 4.141, 4.142, 4.143, 4.181

Solutions will be discussed this Friday by the TA.