STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 15: Multivariate Probability Distributions

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Outline

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2. Previously...
   - Moment Generating Functions
3. Multivariate Probability Distributions
   - Introduction
   - Joint Probability Mass Function (Discrete)
   - Joint Cumulative Distribution Function (Discrete/Continuous)
   - Joint Probability Density Function (Continuous)

Quiz 4 Review Exercises Solutions

Problem 1

Suppose the grades on this quiz is normally distributed with a mean score of 70 points and standard deviation of 10 points. Furthermore, suppose I decide to give the top 10% a bonus of 5 points. What should be the cutoff score to merit the bonus points?

Solution:

- We want to find the cutoff score x such that
$$P(X \geq x) = 0.10.$$
- This is equivalent to finding the cutoff score x such that
$$\begin{aligned} P\left(\frac{X-\mu}{\sigma} \geq \frac{x-\mu}{\sigma}\right) &= 0.10 \\ \Rightarrow P\left(Z \geq \frac{x-\mu}{\sigma}\right) &= 0.10 \\ \Rightarrow 1 - P\left(Z < \frac{x-\mu}{\sigma}\right) &= 0.10 \\ \Rightarrow P\left(Z < \frac{x-\mu}{\sigma}\right) &= 0.90. \end{aligned}$$

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Problem 1

Suppose the grades on this quiz is normally distributed with a mean score of 70 points and standard deviation of 10 points. Furthermore, suppose I decide to give the top 10% a bonus of 5 points. What should be the cutoff score to merit the bonus points?

Solution:

- From the Z-table, $P(Z < 1.28) \approx 0.9$. This means that

$$\frac{x-\mu}{\sigma} = 1.28.$$

  Solving for x and replacing $\mu = 70$ and $\sigma = 10$ (given), we have

$$\begin{aligned} \frac{x-70}{10} &= 1.28 \\ \Rightarrow x - 70 &= 12.8 \\ \Rightarrow x &= 82.8. \end{aligned}$$

- Thus, the cutoff score for the bonus points is 82.8.

Problem 2

Let X have MGF given by

$$m(t) = \frac{1}{3}e^t + \frac{2}{3}e^{2t}, \quad t \in \mathbb{R}.$$

(a) What is the distribution of X?

Solution:

- Matching the MGF above to the MGF formula $m(t) = E(e^{tX}) = \sum_y e^{tx} p(x)$, we know that the MGF above corresponds to a discrete random variable with PMF:

$$p(x) = \begin{cases} \frac{1}{3}, & \text{if } x = 1, \\ \frac{2}{3}, & \text{if } x = 2, \\ 0, & \text{elsewhere.} \end{cases}$$

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Problem 2

Let X have MGF given by

$$m(t) = \frac{1}{3}e^{t} + \frac{2}{3}e^{2t}, \quad t \in \mathbb{R}.$$

(b) Find the expected value and variance of X.

Solution:

- Finding the expected value:

$$\begin{aligned} m'(t) &= \frac{1}{3}e^{t} + \frac{4}{3}e^{2t} \\ E(X) = m'(0) &= \frac{1}{3}e^{(0)} + \frac{4}{3}e^{2(0)} = \frac{5}{3}. \end{aligned}$$

- Finding the variance:

$$\begin{aligned} m''(t) &= \frac{1}{3}e^{t} + \frac{8}{3}e^{2t} \\ E(X^2) = m''(0) &= \frac{1}{3}e^{(0)} + \frac{8}{3}e^{2(0)} = \frac{9}{3} = 3. \\ V(X) &= E(X^2) - \{E(X)\}^2 = 3 - \left(\frac{5}{3}\right)^2 = \frac{2}{9}. \end{aligned}$$

□

Problem 3

Verify that the standard normal PDF

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty,$$

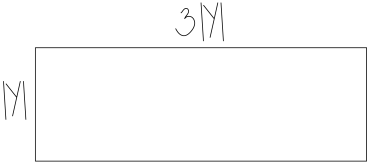
is a valid PDF.

Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(z)dz &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \quad \text{since } \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \text{ is an even function} \\
&= 2\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} z^{-1} e^{-\frac{z^2}{2}} z dz \quad \text{multiply a factor of 1} \\
&= 2\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (\sqrt{2t})^{-1} e^{-t} dt \quad \text{Let } t = z^2/2 \Rightarrow dt = zdz. \Rightarrow z = \sqrt{2t}. \\
&= 2\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{0}^{\infty} t^{-\frac{1}{2}} e^{-t} dt = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} t^{-\frac{1}{2}} e^{-t} dt = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{1}{2}-1} e^{-t} dt \\
&= \frac{1}{\sqrt{\pi}} \Gamma(1/2) \quad \text{Recall the Gamma function: } \Gamma(\alpha) = \int_{0}^{\infty} y^{\alpha-1} e^{-y} dy. \\
&= \frac{1}{\sqrt{\pi}} \sqrt{\pi} \quad \text{Recall properties of the Gamma function: } \Gamma(1/2) = \sqrt{\pi}. \\
&= 1. \quad \text{Thus, the standard normal PDF is a valid PDF.} \quad \square
\end{aligned}$$

Problem 4

Suppose $Y \sim \mathcal{N}(\mu, \sigma^2)$. Find the expected value of the area of the rectangle below.



Solution:

- Given: $Y \sim \mathcal{N}(\mu, \sigma^2)$, $L = 3|Y|$, $W = |Y|$.
- Formula for area of rectangle: $A = L \times W$.
- Thus,

$$\begin{aligned} E(A) &= E(3|Y| \times |Y|) \\ &= 3E(Y^2) \quad \text{linearity property of expectation} \\ &= 3[V(Y) + \{E(Y)\}^2] \quad \text{variance formula} \\ &= 3(\sigma^2 + \mu^2). \quad \text{given} \quad \square \end{aligned}$$

Problem 5

Suppose that X has the Gamma distribution with parameters α and β. Let c be a positive constant. Show that cX has the Gamma distribution with parameters α and $c\beta$.

Solution:

We can use the MGF to solve this problem. The MGF of cX is

$$\begin{aligned} m_{cX}(t) &= E\left(e^{tcX}\right) && \text{def'n of MGF} \\ &= E\left(e^{(tc)X}\right) \\ &= m_X(ct) && \text{def'n of MGF} \quad \text{isolate the random variable } X \\ &= \frac{1}{(1-\beta ct)^{\alpha}}. && \text{Since } X \sim \text{Gam}(\alpha,\beta)\text{, MGF of Gamma: } m(t) = \frac{1}{(1-\beta t)^{\alpha}}. \end{aligned}$$

Here ct is used instead of t.

The MGF above is identical to the MGF of a Gamma distribution with parameters α and $c\beta$. Thus, $cX \sim \text{Gam}(\alpha, c\beta)$. □

Previously...

Moment Generating Functions

- *k*th Moment: $E(Y^k) = \mu_k^{'}$
- *k*th Central Moment: $E\{(Y - \mu)^k\} = \mu_k$
- Moment Generating Function (MGF): $m(t) = E\left(e^{tY}\right)$
- To obtain the *k*th moment:

$$E(Y^k) = \mu_k^{'} = \frac{d^k m(t)}{dt^k}\bigg|_{t=0} = m_Y^{(k)}(0).$$

- MGF of a linear transformation: $m_{aX+b}(t) = e^{bt} m_X(at)$

Moment Generating Functions

Distribution	PMF/PDF	E(Y)	V(Y)	MGF
Bernoulli	$p(y) = p^y(1-p)^{1-y}$	p	$p(1-p)$	$m(t) = pe^t + 1 - p$
Binomial	$p(y) = \binom{n}{y} p^y(1-p)^{n-y}$	np	$np(1-p)$	$m(t) = (pe^t + 1 - p)^n$
Geometric	$p(y) = (1-p)^{y-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$m(t) = pe^t \frac{1}{1-e^t(1-p)}$
Poisson	$p(y) = \frac{\lambda^y}{y!}e^{-\lambda}$	λ	λ	$m(t) = e^{\lambda(e^t-1)}$
Uniform	$f(y) = \frac{1}{\theta_2-\theta_1}$	$\frac{\theta_1+\theta_2}{2}$	$\frac{(\theta_2-\theta_1)^2}{12}$	$m(t) = \begin{cases} \frac{e^{t\theta_2}-e^{t\theta_1}}{t(\theta_2-\theta_1)}, & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$
Std. Normal	$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$	0	1	$m(t) = e^{\frac{t^2}{2}}$
Normal	$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}$	μ	σ^2	$m(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$
Exponential	$f(y) = \frac{1}{\beta}e^{-y/\beta}$	β	β^2	$m(t) = \frac{1}{1-\beta t}$
Gamma	$f(y) = \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}$	$\alpha\beta$	$\alpha\beta^2$	$m(t) = \frac{1}{(1-\beta t)^\alpha}$

Multivariate Probability Distributions

We live in a multivariate world...

NORMAL VITAL SIGNS IN ADULTS

CORE TEMPERATURE	98.6°F (37°C)
HEART RATE	60–100 beats per minute
RESPIRATORY RATE	12–18 breaths per minute
BLOOD OXYGEN	95–100%
BLOOD PRESSURE	120/80 mm Hg

healthline

Illustration by Wenzdai Figueroa

We live in a multivariate world...

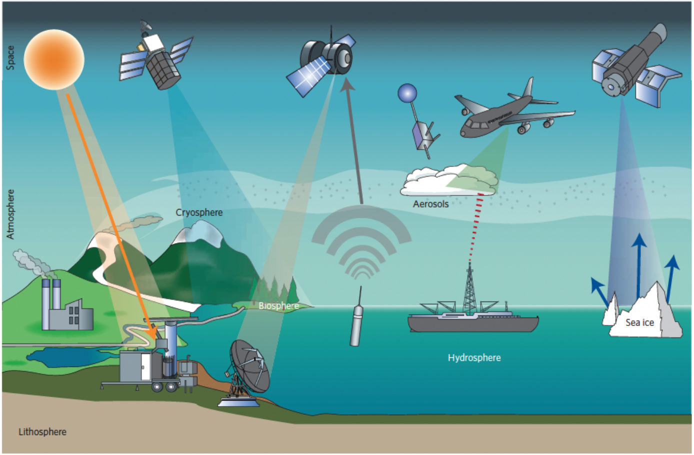


Figure 1 | Remote sensing of the climate system. Remote sensing is carried out by sensors aboard different platforms, including plane, boat and Argo floats. Ground-based instruments are also used, for example, sun spectral radiometers measure solar radiation. However, satellite remote sensing is capable of providing more frequent and repetitive coverage over a large area than other observation means . Figure courtesy of R. He, Hainan University.

Source: https://doi.org/10.1038/nclimate1908

Multivariate Probability Distributions: Introduction

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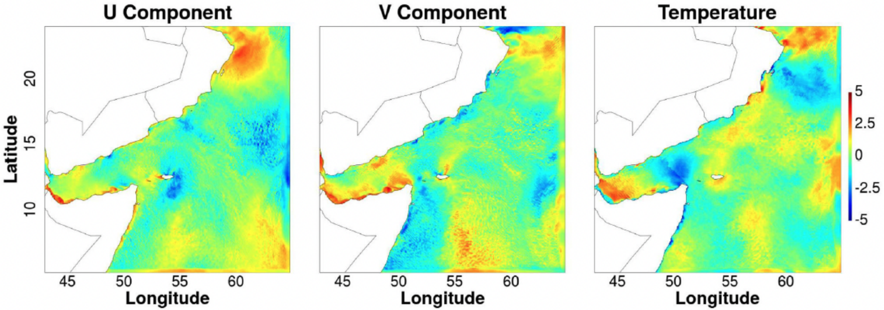

Fig. 17. Spatial images of the trivariate dataset on October 1, 2009 (after mean removal) on 116, 100 locations over the Arabian Sea.

Source: Salvaña, M. L. O., Abdulah, S., Huang, H., Ltaief, H., Sun, Y., Genton, M. G., & Keyes, D. E. (2021). High performance multivariate spatial modeling for geostatistical data on manycore systems. IEEE Transactions on Parallel and Distributed Systems, 32(11), 2719-2733.

Multivariate Probability Distributions: Introduction

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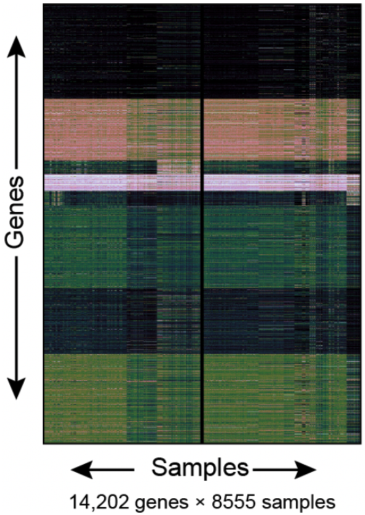
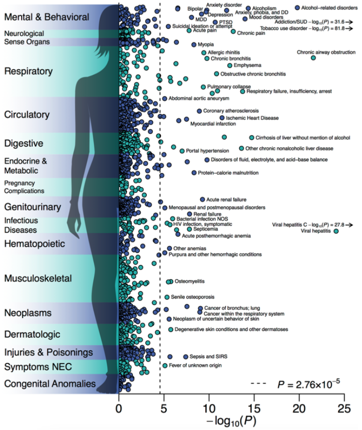


Source: https://doi.org/10.1101/091926

Multivariate Probability Distributions: Introduction

We live in a multivariate world...

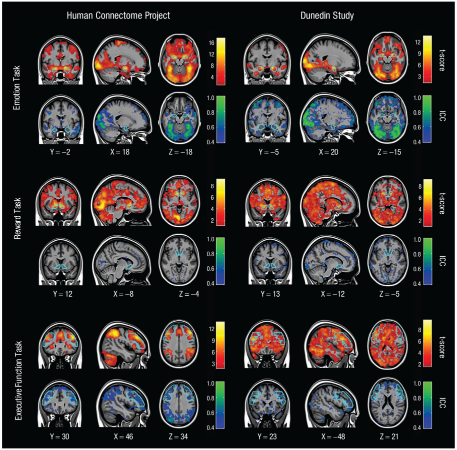


Analyses of genetic correlations among externalizing traits.

Source: https://doi.org/10.1101/2020.10.16.342501

Multivariate Probability Distributions: Introduction

We live in a multivariate world...

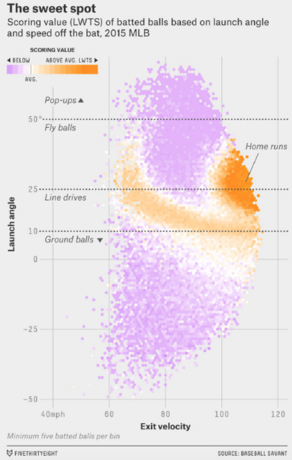


fMRI data showing how different tasks activate different nodes of the brain.

Source: https://doi.org/10.1177/0956797620916786

Multivariate Probability Distributions: Introduction

We live in a multivariate world...

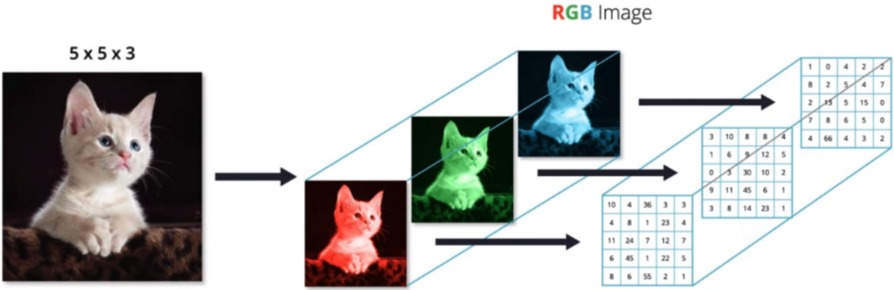


The plot reveals the right trade off between ball velocity and angle.

Source: https://fivethirtyeight.com/features/the-new-science-of-hitting/

Multivariate Probability Distributions: Introduction

We live in a multivariate world...



Colored images can be represented by three color channels namely red, green and blue.

Source: https://dev.to/sandeepbalachandran/machine-learning-going-furthur-with-cnn-part-2-41km

Multivariate Probability Distributions: Introduction

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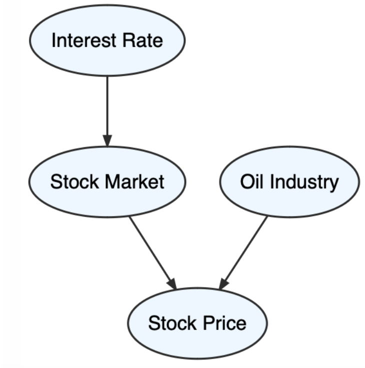

A graphical model of how stock prices are influenced by other factors.

Source: https://www.causact.com/joint-distributions-tell-you-everything

Multivariate Probability Distributions: Introduction

We live in a multivariate world...

Source: https://centerpointsecurities.com/interconnectedness-of-markets/

Multivariate Probability Distributions: Introduction

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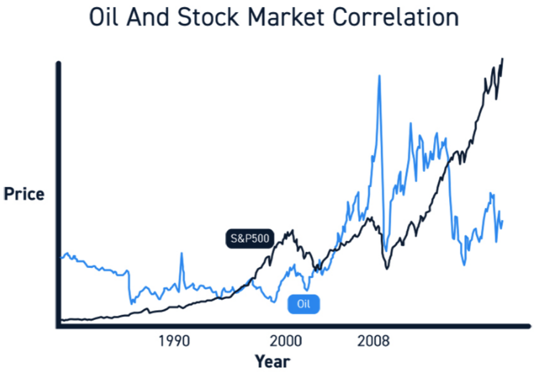

Source: https://centerpointsecurities.com/interconnectedness-of-markets/

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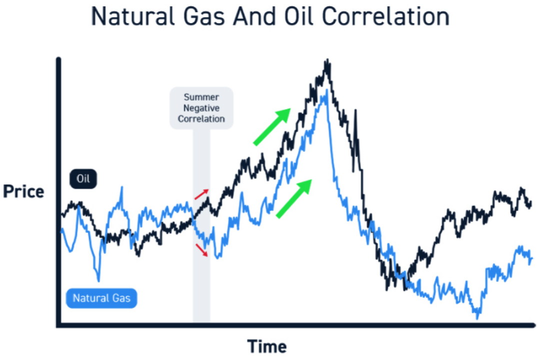

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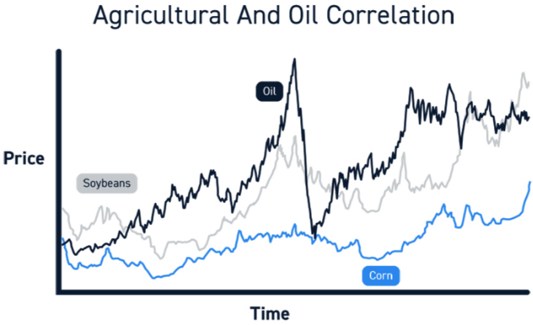

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Multivariate Probability Distributions: Introduction

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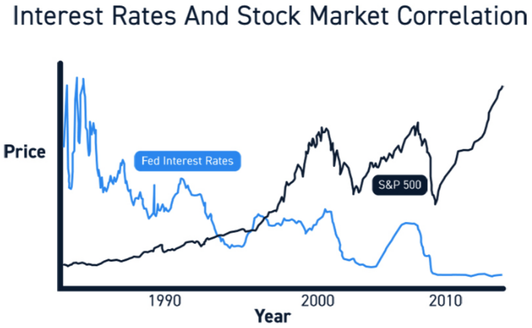


Source: https://centerpointsecurities.com/interconnectedness-of-markets/

Multivariate Probability Distributions: Introduction

	univariate		bivariate		trivariate				multivariate
PMF	$p(y)$	$\Rightarrow$	$p(y_1,y_2)$	$\Rightarrow$	$p(y_1,y_2,y_3)$	$\Rightarrow$	$\cdots$	$\Rightarrow$	$p(y_1,\ldots,y_n)$
PDF	$f(y)$	$\Rightarrow$	$f(y_1,y_2)$	$\Rightarrow$	$f(y_1,y_2,y_3)$	$\Rightarrow$	$\cdots$	$\Rightarrow$	$f(y_1,\ldots,y_n)$
CDF	$F(y)$	$\Rightarrow$	$F(y_1,y_2)$	$\Rightarrow$	$F(y_1,y_2,y_3)$	$\Rightarrow$	$\cdots$	$\Rightarrow$	$F(y_1,\ldots,y_n)$

Recall the Univariate Gaussian PDF:

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

Multivariate challenge: How will this look like with two random variables?

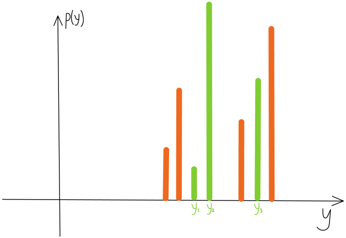
Bivariate Gaussian PDF:

$$f(y_1, y_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{y_1-\mu_1}{\sigma_1}\right)\left(\frac{y_2-\mu_2}{\sigma_2}\right) + \left(\frac{y_2-\mu_2}{\sigma_2}\right)^2\right\}},$$

- μ_1 and σ_1^2 are the mean and variance, respectively, of Y_1,
- μ_2 and σ_2^2 are the mean and variance, respectively, of Y_2,
- $\sigma_{12} = \rho\sigma_1\sigma_2$ is the covariance of Y_1 and Y_2, where ρ is the correlation coefficient

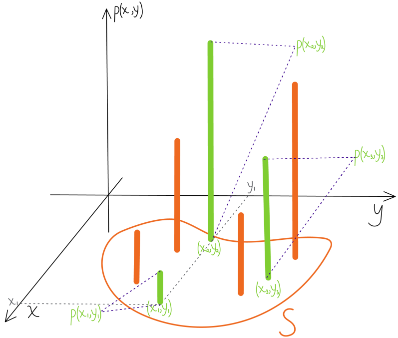
Univariate → Bivariate (Discrete)

$A=\{y_1, y_2, y_3\}$



$$P(A)=p(y_1)+p(y_2)+p(y_3)$$

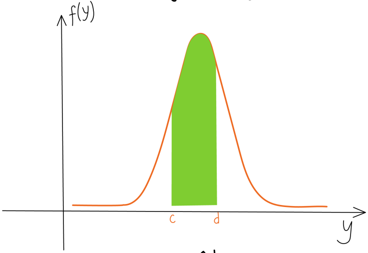
$A=\{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$



$$P(A)=p(x_1,y_1)+p(x_2,y_2)+p(x_3,y_3)$$

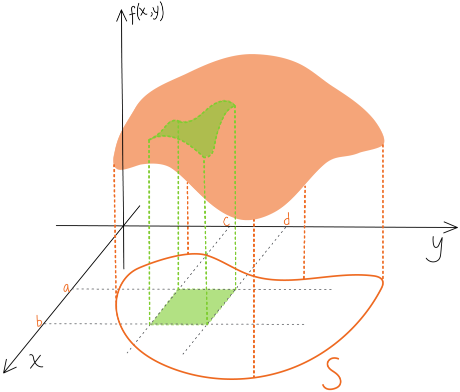
Univariate → Bivariate (Continuous)

$$A = \{c \le Y \le d\}$$



$$P(A) = \int_c^d f(y)\,dy$$

$$A = \{a \le X \le b, c \le Y \le d\}$$



$$P(A) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Joint Probability Mass Function (Discrete)

Definition: Joint PMF for Discrete Random Variables

Let $Y_1, Y_2, \ldots, Y_n$ be discrete random variables. The *joint probability mass function (PMF)* for $Y_1, Y_2, \ldots, Y_n$ is given by

$$p(y_1, y_2, \ldots, y_n) = P(Y_1 = y_1, Y_2 = y_2, \cdots, Y_n = y_n),$$

for $-\infty < y_1, y_2, \ldots, y_n < \infty$.

- $p(y_1, y_2, \ldots, y_n)$ gives the probability of the following event:

$$\{Y_1 = y_1\} \cap \{Y_2 = y_2\} \cap \ldots \cap \{Y_n = y_n\}.$$

- The joint PMF can be summarized/described by a table.

	$X \backslash Y$	y 1	2	3	4
x	1	$P(X=1, Y=1)$	$P(X=1, Y=2)$	$P(X=1, Y=3)$	$P(X=1, Y=4)$
	2	$P(X=2, Y=1)$	$P(X=2, Y=2)$	$P(X=2, Y=3)$	$P(X=2, Y=4)$
	3	$P(X=3, Y=1)$	$P(X=3, Y=2)$	$P(X=3, Y=3)$	$P(X=3, Y=4)$
	4	$P(X=4, Y=1)$	$P(X=4, Y=2)$	$P(X=4, Y=3)$	$P(X=4, Y=4)$

Joint Probability Mass Function (Discrete)

Example 1:

Roll two dice. Let X and Y be the value on the first and second die, respectively. Write the joint probability table of X and Y.

Solution:

- The sample space of X and Y is $\{1, 2, 3, 4, 5, 6\}$.
  Thus, let $x, y = 1, 2, 3, 4, 5, 6$.
- We know that
  $P(X = x, Y = y) = P(X = x)P(Y = y) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$.
- Thus, the joint probability table of X and Y is as follows:

		y					
	$X \backslash Y$	1	2	3	4	5	6
x	1	1/36	1/36	1/36	1/36	1/36	1/36
	2	1/36	1/36	1/36	1/36	1/36	1/36
	3	1/36	1/36	1/36	1/36	1/36	1/36
	4	1/36	1/36	1/36	1/36	1/36	1/36
	5	1/36	1/36	1/36	1/36	1/36	1/36
	6	1/36	1/36	1/36	1/36	1/36	1/36

Example 2:

Roll two dice. Let X be the value on the first die and let T be the total on both dice. Write the joint probability table of X and T.

Solution:

- Let $x = 1, 2, 3, 4, 5, 6$ and $t = 2, 3, \ldots, 12$.
- Let Y be the value on the second die and Y takes on values $1 \leq y \leq 6$.

$$\begin{aligned} P(X = x, T = t) &= P(X = x, Y = t - x) = P(X = x)P(Y = t - x) \\ &= \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}, \text{ for } 1 \leq t - x \leq 6 \Rightarrow 1 + x \leq t \leq 6 + x. \end{aligned}$$

- Thus, the joint probability table of X and T is as follows:

	$X \backslash T$	t: 2	3	4	5	6	7	8	9	10	11	12
x	1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0
	2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0
	3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0
	4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0
	5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0
	6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36

Joint Probability Mass Function (Discrete)

Theorem: Properties of Joint PMFs

If $Y_1, Y_2, \ldots, Y_n$ are discrete random variables with joint PMF $p(y_1, y_2, \ldots, y_n)$, then

1. $p(y_1, y_2, \ldots, y_n) \geq 0$, for all $y_1, y_2, \ldots, y_n$.
2. $\sum_{y_1, y_2, \ldots, y_n} p(y_1, y_2, \ldots, y_n) = 1$, where the sum is over all values $(y_1, y_2, \ldots, y_n)$ that are assigned nonzero probabilities.

Joint Probability Mass Function (Discrete)

Example 3: Consider X, Y with the following joint PMF $p(x, y)$:

	$X \backslash Y$	y 1	2	3	4
x	1	1/16	0	1/8	1/16
	2	1/32	1/32	1/4	0
	3	0	1/8	1/16	1/16
	4	1/16	1/32	1/16	1/32

a. Is the PMF above valid?

Solution:

Check that the following properties are satisfied:

- $p(x, y) \geq 0$ for all x and y.
- $\sum_{x=1}^{4} \sum_{y=1}^{4} p(x, y) = 1$.

Joint Probability Mass Function (Discrete)

Example 3: Consider X, Y with the following joint PMF $p(x, y)$:

		y			
	$X \backslash Y$	1	2	3	4
x	1	1/16	0	1/8	1/16
	2	1/32	1/32	1/4	0
	3	0	1/8	1/16	1/16
	4	1/16	1/32	1/16	1/32

(b) Find $P(X = Y)$.

Solution:

$$\begin{aligned} P(X = Y) &= p(1,1) + p(2,2) + p(3,3) + p(4,4) \\ &= 1/16 + 1/32 + 1/16 + 1/32 \\ &= 0.1875. \end{aligned}$$

Joint Probability Mass Function (Discrete)

Example 4: The joint distribution $p(x, y)$ of X (number of cars) and Y (number of buses) per signal cycle at a traffic signal is given by:

	$X \setminus Y$	y: 0	1	2
	0	0.025	0.015	0.010
	1	0.050	0.030	0.020
x	2	0.125	0.075	0.050
	3	0.150	0.090	0.060
	4	0.100	0.060	0.040
	5	0.050	0.030	0.020

a. Is the PMF above valid?

Solution:

Check that the following properties are satisfied:

- $p(x, y) \geq 0$ for all x and y.
- $\sum_{x=0}^{5} \sum_{y=0}^{2} p(x, y) = 1$.

Joint Probability Mass Function (Discrete)

Example 4: The joint distribution $p(x, y)$ of X (number of cars) and Y (number of buses) per signal cycle at a traffic signal is given by:

		y		
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	3	0.150	0.090	0.060
	4	0.100	0.060	0.040
	5	0.050	0.030	0.020

(b) Find $P(X = Y)$.

Solution:

$$\begin{aligned} P(X = Y) &= p(0,0) + p(1,1) + p(2,2) \\ &= 0.025 + 0.030 + 0.050 \\ &= 0.105. \end{aligned}$$

Joint Probability Mass Function (Discrete)

Example 5: Let X be a coin flip and Y be a die. Find the joint PMF.

Solution:

- The sample space of X is $\{0, 1\}$.
- The sample space of Y is $\{1, 2, 3, 4, 5, 6\}$.
- We know that $P(X = x, Y = y) = P(X = x)P(Y = y) = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{12}$.
- The joint PMF therefore is

	$X \backslash Y$	y 1	2	3	4	5	6
x	0	1/12	1/12	1/12	1/12	1/12	1/12
	1	1/12	1/12	1/12	1/12	1/12	1/12

- Or written as an equation:

$$p(x, y) = \frac{1}{12}, \quad x = 0, 1, \quad y = 1, 2, 3, 4, 5, 6.$$

Joint Probability Mass Function (Discrete)

Example 6: Let X be a coin flip and Y be a die.
Define $A = \{X + Y = 3\}$. Find $P(A)$.

Solution:

Recall the joint probability table from previous slide:

	$X \backslash Y$	y 1	2	3	4	5	6
x	0	1/12	1/12	1/12	1/12	1/12	1/12
	1	1/12	1/12	1/12	1/12	1/12	1/12

$$
\begin{aligned}
P(A) &= \sum_{(x,y)\in A} p(x,y) \\
&= p(0,3) + p(1,2) \\
&= \frac{1}{12} + \frac{1}{12} \\
&= \frac{1}{6}.
\end{aligned}
$$

Joint Probability Mass Function (Discrete)

Example 7: Let X be a coin flip and Y be a die.
Define $B = \{\min(X, Y) = 1\}$. Find $P(B)$.
Solution:
Recall the joint probability table from previous slide:

	$X \backslash Y$	y 1	2	3	4	5	6
x	0	1/12	1/12	1/12	1/12	1/12	1/12
	1	1/12	1/12	1/12	1/12	1/12	1/12

$$\begin{aligned} P(B) &= \sum_{(x,y)\in B} p(x,y) \\ &= p(1,1) + p(1,2) + p(1,3) + p(1,4) + p(1,5) + p(1,6) \\ &= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} \\ &= \frac{1}{2}. \end{aligned}$$

Joint Cumulative Distribution Function (Disc./Cont.)

Definition: Joint CDF

Let $Y_1, Y_2, \ldots, Y_n$ be (discrete or continuous) random variables. The *joint cumulative distribution function* (CDF) for $Y_1, Y_2, \ldots, Y_n$ is

$$F(y_1, y_2, \ldots, y_n) = P(Y_1 \leq y_1, Y_2 \leq y_2, \cdots, Y_n \leq y_n),$$

for $-\infty < y_1, y_2, \ldots, y_n < \infty$.

- $F(y_1, y_2, \ldots, y_n)$ gives the probability of the following event:

$$\{Y_1 \leq y_1\} \cap \{Y_2 \leq y_2\} \cap \ldots \cap \{Y_n \leq y_n\}.$$

Joint Cumulative Distribution Function (Disc./Cont.)

Theorem: Properties of Joint CDFs

1. If $Y_1, Y_2, \ldots, Y_n$ are random variables with joint CDF $F(y_1, y_2, \ldots, y_n)$, then

$$F(-\infty, -\infty, \ldots, -\infty) = F(y_1, -\infty, \ldots, -\infty) = F(-\infty, y_2, \ldots, -\infty) = \cdots = F(-\infty, -\infty, \ldots, y_n) = 0.$$

2. $F(\infty, \infty, \ldots, \infty) = 1.$

- Condition 1 tells us that

$$\begin{aligned} & P(\{Y_1 \leq -\infty\} \cap \{Y_2 \leq -\infty\} \cap \ldots \cap \{Y_n \leq -\infty\}) \\ = \; & P(\{Y_1 \leq -\infty\} \cap \{Y_2 \leq y_2\} \cap \ldots \cap \{Y_n \leq -\infty\}) \\ & \vdots \\ = \; & P(\{Y_1 \leq -\infty\} \cap \{Y_2 \leq -\infty\} \cap \ldots \cap \{Y_n \leq y_n\}) = 0. \end{aligned}$$

- Condition 2 tells us that

$$P(\{Y_1 \leq \infty\} \cap \{Y_2 \leq \infty\} \cap \ldots \cap \{Y_n \leq \infty\}) = 1.$$

Joint Cumulative Distribution Function (Discrete)

Example 8: Recall the joint probability for X, Y in Example 1. Compute $F(3.5, 4)$.

Solution:

- $F(3.5, 4) = P(X \leq 3.5, Y \leq 4)$. We can visualize this event as the shaded cells in the table:

	$X \backslash Y$	y 1	2	3	4	5	6
x	1	1/36	1/36	1/36	1/36	1/36	1/36
	2	1/36	1/36	1/36	1/36	1/36	1/36
	3	1/36	1/36	1/36	1/36	1/36	1/36
	4	1/36	1/36	1/36	1/36	1/36	1/36
	5	1/36	1/36	1/36	1/36	1/36	1/36
	6	1/36	1/36	1/36	1/36	1/36	1/36

- Adding up the probabilities, we get $F(3.5, 4) = 12 \times \frac{1}{36} = \frac{1}{3}$.

Joint Probability Density Function (Continuous)

Definition: Joint PDF for Continuous Random Variables

Let $Y_1, Y_2, \ldots, Y_n$ be continuous random variables with joint CDF $F(y_1, y_2, \ldots, y_n)$. If there exists a nonnegative function $f(y_1, y_2, \ldots, y_n)$, such that

$$F(y_1, y_2, \ldots, y_n) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_n} f(t_1, t_2, \ldots, t_n) dt_1 dt_2 \cdots dt_n,$$

for $-\infty < y_1, y_2, \ldots, y_n < \infty$, then $Y_1, Y_2, \ldots, Y_n$ are said to be *jointly continuous random variables*. The function $f(y_1, y_2, \ldots, y_n)$ is called the *joint probability density function (PDF)*.

Joint Probability Density Function (Continuous)

Theorem: Properties of Joint PDFs

If $f(y_1, y_2, \ldots, y_n)$ is a joint density function for $Y_1, Y_2, \ldots, Y_n$, then

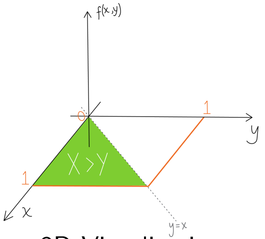
1. $f(y_1, y_2, \ldots, y_n) \geq 0$, for all $y_1, y_2, \ldots, y_n$.
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y_1, y_2, \ldots, y_n) dy_1 dy_2 \cdots dy_n = 1$.

Example 9:

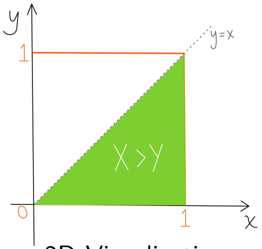
Suppose X and Y both take values in $[0,1]$ with uniform density $f(x,y)=1$. Visualize the event $X>Y$ and find its probability.

Solution:

- The event $X>Y$ corresponds to the region in the unit square where the x values are greater than the y values.
- Use the line $y=x$ as a guide to find the region corresponding to the event $X>Y$.



3D Visualization



2D Visualization

- $P(X>Y)=\int\int_{\{(x,y):X>Y\}} f(x,y)dxdy=\int_0^1\int_y^1 1dxdy=\int_0^1 x\big|_y^1 dy=$ $\int_0^1(1-y)dy=\left(y-\frac{y^2}{2}\right)\big|_0^1=1-\frac{1}{2}=\frac{1}{2}$.

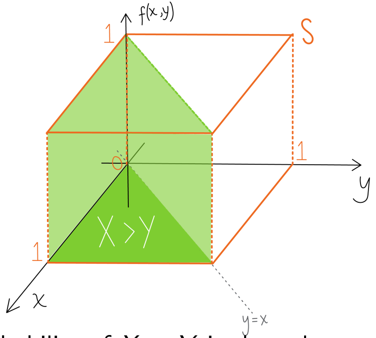
Joint Probability Density Function (Continuous)

Example 9:

Suppose X and Y both take values in $[0, 1]$ with uniform density $f(x, y) = 1$. Visualize the event $X > Y$ and find its probability.

Solution:

- Another way to compute $P(X > Y)$:



Probability of $X > Y$ is the volume of the triangular prism.

$$P(X > Y) = \frac{1}{2} \times \text{volume of cube} = \frac{1}{2}. \quad \text{Vol. cube} = 1$$

OR

$$P(X > Y) = \text{vol. of triangular prism} = \text{Area of base} \times \text{height} = \frac{1}{2}. \quad \text{Area of base} = 1/2,\ \text{height} = 1$$

Joint Probability Density Function (Continuous)

Example 10:

Suppose X and Y have the joint PDF:

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x+y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Is this is a valid PDF?

Solution:

- $f(x,y)$ is nonnegative everywhere.
- We need to check whether $f(x,y)$ integrates to 1.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy \quad = \quad \int_{???}^{???}\int_{???}^{???} 6x^2y\,dxdy$$

- KEY QUESTION: What are the bounds of integration?
  (cont'd next slide...)

Joint Probability Density Function (Continuous)

Example 10:

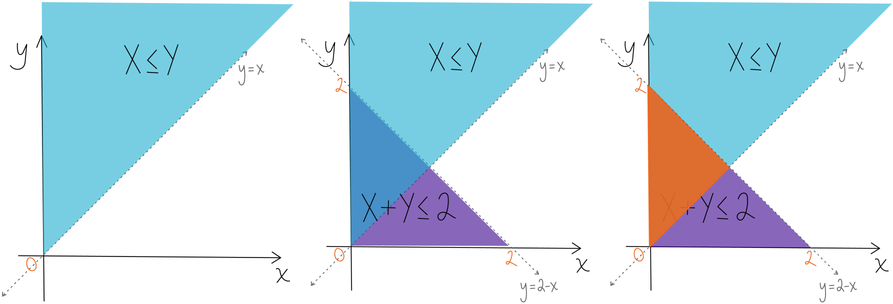
Suppose X and Y have the joint PDF:

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x + y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Is this is a valid PDF?

Solution:

- KEY QUESTION: What are the bounds of integration?



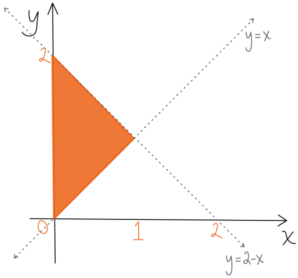
Example 10:

Suppose X and Y have the joint PDF:

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x+y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Is this is a valid PDF? Check whether $\int\int f(x,y)dxdy = 1$.

Solution:



- $\int_0^1 \int_x^{2-x} 6x^2 y dy dx = \int_0^1 6x^2 \frac{y^2}{2}\Big|_x^{2-x} dx$

$$= \int_0^1 6x^2 \frac{(2-x)^2 - x^2}{2} dx$$
$$= \int_0^1 6x^2 \frac{4-4x+x^2-x^2}{2} dx$$
$$= \int_0^1 6x^2 (2-2x) dx$$
$$= \int_0^1 6x^2 2(1-x) dx$$
$$= 12 \int_0^1 x^2 (1-x) dx$$

Beta function: $B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$= 12B(3,2) = 12\frac{\Gamma(3)\Gamma(2)}{\Gamma(5)}$$
$$= 12\frac{2!1!}{4!} = 1.$$

□

Joint Probability Density Function (Continuous)

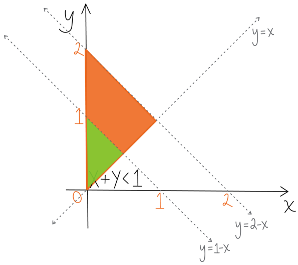
Example 10:

Suppose X and Y have the joint PDF:

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x+y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(b) What is the probability that $X + Y$ is less than 1?

Solution:



- $$\begin{aligned} P(X+Y<1) &= \int_0^{1/2}\int_x^{1-x} 6x^2 y\,dy\,dx \\ &= \int_0^{1/2} 6x^2 \frac{y^2}{2}\Big|_x^{1-x} dx \\ &= \int_0^{1/2} 6x^2 \frac{(1-x)^2 - x^2}{2} dx \\ &= \int_0^{1/2} 6x^2 \frac{1-2x+x^2-x^2}{2} dx \\ &= \int_0^{1/2} 3x^2(1-2x)dx \\ &= \int_0^{1/2} 3x^2 - 6x^3 dx \\ &= x^3 - \frac{6}{4}x^4 \Big|_0^{1/2} = \frac{1}{8} - \frac{3}{2}\left(\frac{1}{2}\right)^4 \\ &= \frac{1}{8} - \frac{3}{32} = \frac{1}{32}. \end{aligned}$$

□

Questions?

Homework Exercises: 4.139, 4.141, 4.142, 4.143, 4.181

Solutions will be discussed this Friday by the TA.