STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 16: Marginal and Conditional Probability Distributions; Independent Random Variables

Mary Lai Salvaña, Ph.D.

Department of Statistics
University of Connecticut

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Outline

Previously...

Multivariate Probability Distributions

- Multivariate extensions of univariate concepts:

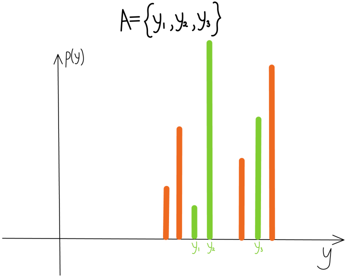
	univariate		bivariate		trivariate				multivariate
PMF	$p(y)$	$\Longrightarrow$	$p(y_1,y_2)$	$\Longrightarrow$	$p(y_1,y_2,y_3)$	$\Longrightarrow$	$\cdots$	$\Longrightarrow$	$p(y_1,\ldots,y_n)$
PDF	$f(y)$	$\Longrightarrow$	$f(y_1,y_2)$	$\Longrightarrow$	$f(y_1,y_2,y_3)$	$\Longrightarrow$	$\cdots$	$\Longrightarrow$	$f(y_1,\ldots,y_n)$
CDF	$F(y)$	$\Longrightarrow$	$F(y_1,y_2)$	$\Longrightarrow$	$F(y_1,y_2,y_3)$	$\Longrightarrow$	$\cdots$	$\Longrightarrow$	$F(y_1,\ldots,y_n)$

- Joint PMF (Discrete):
  $p(y_1, y_2, \ldots, y_n) = P(Y_1 = y_1, Y_2 = y_2, \cdots, Y_n = y_n)$
- Joint CDF (Discrete/Continuous):
  $F(y_1, y_2, \ldots, y_n) = P(Y_1 \leq y_1, Y_2 \leq y_2, \cdots, Y_n \leq y_n)$
- Joint PDF (Continuous):
  $F(y_1, y_2, \ldots, y_n) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_n} f(t_1, t_2, \ldots, t_n) dt_1 dt_2 \cdots dt_n,$

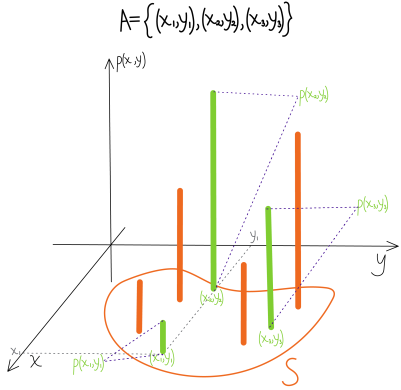
Multivariate Probability Distributions

Univariate → Bivariate (Discrete)

Distribution: From 2D to 3D point mass...



$$P(A)=p(y_1)+p(y_2)+p(y_3)$$



$$P(A)=p(x_1,y_1)+p(x_2,y_2)+p(x_3,y_3)$$

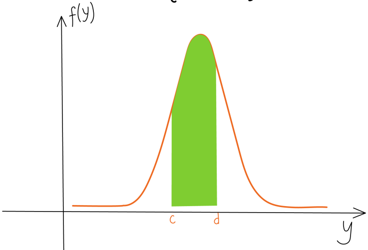
Multivariate Probability Distributions

Univariate → Bivariate (Continuous)

Density: From a 2D curve to a 3D surface...

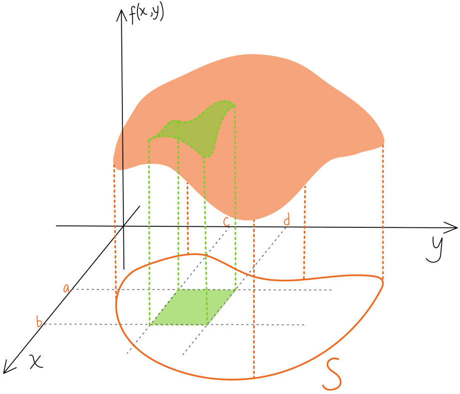
Probability: From area to volume...

$$A = \{c \le Y \le d\}$$



$$P(A) = \int_c^d f(y)\,dy$$

$$A = \{a \le X \le b, c \le Y \le d\}$$



$$P(A) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Multivariate Probability Distributions: Joint PMF

The joint PMF can be summarized/described by a table where

- columns: possible values of Y
- rows: possible values of X
- cells: probabilities of every possible combination of X & Y values

Example 3 in Lec 15:

		y			
	$X \backslash Y$	1	2	3	4
x	1	1/16	0	1/8	1/16
	2	1/32	1/32	1/4	0
	3	0	1/8	1/16	1/16
	4	1/16	1/32	1/16	1/32

What is $P(X = Y)$?

$$\begin{aligned} P(X = Y) &= p(1,1) + p(2,2) + p(3,3) + p(4,4) \\ &= 1/16 + 1/32 + 1/16 + 1/32 \\ &= 0.1875. \end{aligned}$$

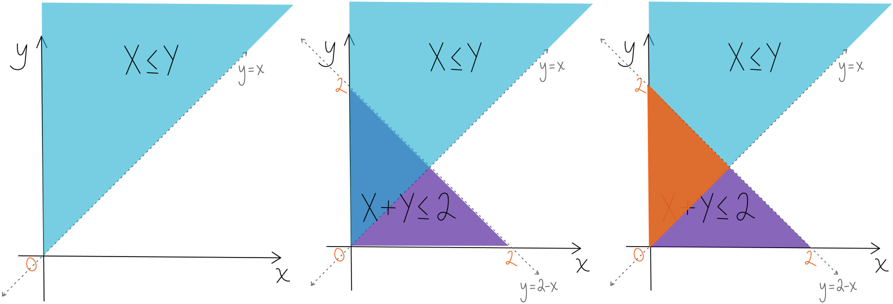
Multivariate Probability Distributions: Joint PDF

Example 10 in Lec 15:

Suppose X and Y have the joint PDF:

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x+y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

KEY QUESTION: What are the bounds of integration?



Multivariate Probability Distributions: Joint PDF

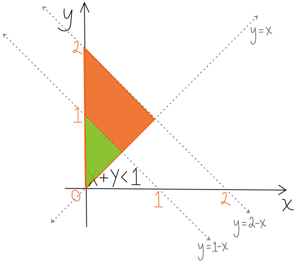
Example 10 in Lec 15:

Suppose X and Y have the joint PDF:

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x+y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

What is the probability that $X+Y$ is less than 1?

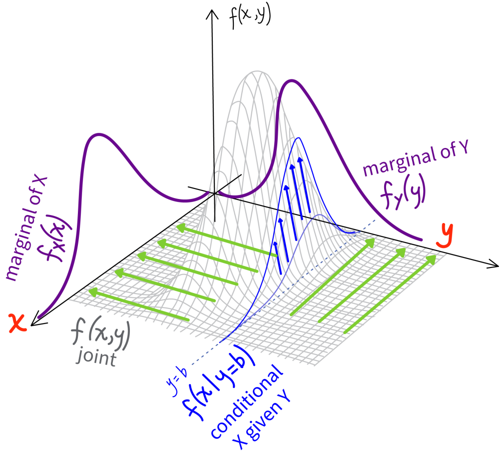
- $$\begin{aligned} P(X+Y<1) &= \int_{???}^{???} \int_{???}^{???} 6x^2y\,dxdy \\ &= \int_0^{1/2} \int_x^{1-x} 6x^2ydydx \\ &= \int_0^{1/2} 6x^2 \frac{y^2}{2}\Big|_x^{1-x} dx \\ &= \int_0^{1/2} 6x^2 \frac{(1-x)^2-x^2}{2} dx \\ &= \int_0^{1/2} 6x^2 \frac{1-2x+x^2-x^2}{2} dx \\ &= \int_0^{1/2} 3x^2(1-2x)dx \\ &= \int_0^{1/2} 3x^2 - 6x^3 dx \\ &= x^3 - \frac{6}{4}x^4 \Big|_0^{1/2} = \frac{1}{8} - \frac{3}{2}\left(\frac{1}{2}\right)^4 \\ &= \frac{1}{8} - \frac{3}{32} = \frac{1}{32}. \end{aligned}$$



□

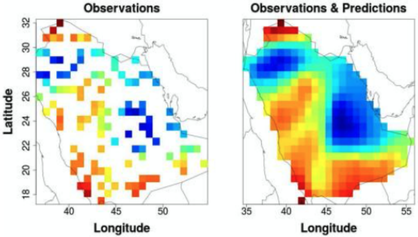
Multivariate Probability Distributions

- 3 types:
  - joint distribution, $f(x, y)$: *probability of both RVs*
  - marginal distributions, $f_X(x)$ and $f_Y(y)$: *probability of RV of interest*
  - conditional distributions, $f(x|y)$ or $f(y|x)$: *updated probability*



Multivariate Probability Distributions: Importance

- Most important task in Statistics: PREDICTION
- For example:
  - Data: temperature measurements at 100 locations
  - Goal: Fill in missing measurements at 400 locations



Observations

Observations & Predictions

  - Step 1: Let $\mathbf{X}$ = temperature measurements at 100 locations.
    Let $\mathbf{Y}$ = missing measurements at 400 locations.
  - Step 2: Assume a joint distribution for $\mathbf{X}$ and $\mathbf{Y}$, $f(\mathbf{x}, \mathbf{y})$.
  - Step 3: Estimate the parameters of the joint distribution using the marginal distribution of $\mathbf{X}$, $f_{\mathbf{X}}(\mathbf{x})$.
  - Step 4: Predict the missing values $\mathbf{Y}$ using the conditional distribution $f(\mathbf{y}|\mathbf{x})$.

Marginal Probability Distributions

Marginal CDF

Definition: Joint CDF → Marginal CDF (Discrete/Continuous)

Let Y_1 and Y_2 be random variables with joint CDF $F(y_1, y_2)$. The *marginal CDF* for Y_1 and Y_2, respectively, are

$$\begin{aligned} F_1(y_1) &= P(Y_1 \leq y_1) = F(y_1, \infty), \quad \text{and} \\ F_2(y_2) &= P(Y_2 \leq y_2) = F(\infty, y_2). \end{aligned}$$

Recall the joint CDF of Y_1 and Y_2: $F(y_1, y_2) = P(\{Y_1 \leq y_1\} \cap \{Y_2 \leq y_2\})$

$$\begin{aligned} \overset{y_2 \to \infty}{\Longrightarrow} F(y_1, \infty) &= P(\{Y_1 \leq y_1\} \cap \{Y_2 \leq \infty\}) \\ &= P(\{Y_1 \leq y_1\} \cap \mathbb{R}) \\ &= P(Y_1 \leq y_1) = F_1(y_1) \end{aligned}$$

$$\begin{aligned} \overset{y_1 \to \infty}{\Longrightarrow} F(\infty, y_2) &= P(\{Y_1 \leq \infty\} \cap \{Y_2 \leq y_2\}) \\ &= P(\mathbb{R} \cap \{Y_2 \leq y_2\}) \\ &= P(Y_2 \leq y_2) = F_2(y_2). \end{aligned}$$

Marginal PMF

Definition: Joint PMF → Marginal PMF (Discrete)

Let Y_1 and Y_2 be jointly discrete random variables with joint PMF $p(y_1, y_2)$. The *marginal PMF* for Y_1 and Y_2, respectively, are

$$p_1(y_1) = P(Y_1 = y_1) = \sum_{\text{all } y_2} p(y_1, y_2)$$

and

$$p_2(y_2) = P(Y_2 = y_2) = \sum_{\text{all } y_1} p(y_1, y_2).$$

- The marginal probability distribution is the probability distribution of one of the two variables – ignoring the other.
- To get the marginal PMF of y_1, sum over y_2.
- To get the marginal PMF of y_2, sum over y_1.

Marginal PMF

Other Remarks:

- The marginal PMF of y_1 is free of y_2.
- The marginal PMF of y_2 is free of y_1.
- A marginal distribution is a compression of information where only information regarding the marginal variable is maintained.
- The process of obtaining the marginal distribution of one variable from the joint distribution is called marginalization.

Marginal PMF: Illustration

How to obtain the marginal PMF of X from this joint PMF of X and Y?

		y			
	$X\backslash Y$	1	2	3	4
x	1	$p(1,1)$	$p(1,2)$	$p(1,3)$	$p(1,4)$
	2	$p(2,1)$	$p(2,2)$	$p(2,3)$	$p(2,4)$
	3	$p(3,1)$	$p(3,2)$	$p(3,3)$	$p(3,4)$
	4	$p(4,1)$	$p(4,2)$	$p(4,3)$	$p(4,4)$

- Create a new column for the marginal PMF of X, $p_X(x)$.

		y				
	$X\backslash Y$	1	2	3	4	$p_X(x)$
x	1	$p(1,1)$	$p(1,2)$	$p(1,3)$	$p(1,4)$	$= p_X(1)$
	2	$p(2,1)$	$p(2,2)$	$p(2,3)$	$p(2,4)$	$= p_X(2)$
	3	$p(3,1)$	$p(3,2)$	$p(3,3)$	$p(3,4)$	$= p_X(3)$
	4	$p(4,1)$	$p(4,2)$	$p(4,3)$	$p(4,4)$	$= p_X(4)$

Marginalization flattens a 2D table to 1D...

Marginal PMF: Illustration

How to obtain the marginal PMF of Y from this joint PMF of X and Y?

		y			
	$X \backslash Y$	1	2	3	4
x	1	$p(1,1)$	$p(1,2)$	$p(1,3)$	$p(1,4)$
	2	$p(2,1)$	$p(2,2)$	$p(2,3)$	$p(2,4)$
	3	$p(3,1)$	$p(3,2)$	$p(3,3)$	$p(3,4)$
	4	$p(4,1)$	$p(4,2)$	$p(4,3)$	$p(4,4)$

- Create a new row for the marginal PMF of Y, $p_Y(y)$.

		y				
	$X \backslash Y$	1	2	3	4	$p_X(x)$
x	1	$p(1,1)$	$p(1,2)$	$p(1,3)$	$p(1,4)$	$= p_X(1)$
	2	$p(2,1)$	$p(2,2)$	$p(2,3)$	$p(2,4)$	$= p_X(2)$
	3	$p(3,1)$	$p(3,2)$	$p(3,3)$	$p(3,4)$	$= p_X(3)$
	4	$p(4,1)$	$p(4,2)$	$p(4,3)$	$p(4,4)$	$= p_X(4)$
	$p_Y(y)$	$= p_Y(1)$	$= p_Y(2)$	$= p_Y(3)$	$= p_Y(4)$	

Marginalization flattens a 2D table to 1D...

Marginal PMF

Example 1: (Recall Ex. 2 in Lec 15)

Roll two dice. Let X be the value on the first die and let T be the total on both dice. Compute the marginal PMFs of X and T.

Solution:

- From Ex. 2 in Lec 15, we have the joint PMF of X and T.

		t										
	$X\backslash T$	2	3	4	5	6	7	8	9	10	11	12
x	1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0
	2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0
	3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0
	4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0
	5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0
	6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36

Marginal PMF

Example 1: (Recall Ex. 2 in Lec 15)

Roll two dice. Let X be the value on the first die and let T be the total on both dice. Compute the marginal PMFs of X and T.

Solution:

- From Ex. 2 in Lec 15, we have the joint PMF of X and T.

		t											
	$X\backslash T$	2	3	4	5	6	7	8	9	10	11	12	$p_X(x)$
x	1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0	1/6
	2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	1/6
	3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	1/6
	4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	1/6
	5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	1/6
	6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	$p_T(t)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	

- The marginal PMF of X is: $p_X(x) = \frac{1}{6}$, $x = 1, 2, \ldots, 6$.
- The marginal PMF of T is: $$p_T(t) = \begin{cases} 1/36, & t = 2, 12 \\ 1/18, & t = 3, 11 \\ 1/12, & t = 4, 10 \\ 1/9, & t = 5, 9 \\ 5/36, & t = 6, 8 \\ 1/6, & t = 7. \end{cases}$$ □

Marginal PDF

Definition: Joint PDF → Marginal PDF (Continuous)

Let Y_1 and Y_2 be jointly continuous random variables with joint PDF $f(y_1, y_2)$. The *marginal PDF* for Y_1 and Y_2, respectively, are

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2$$

and

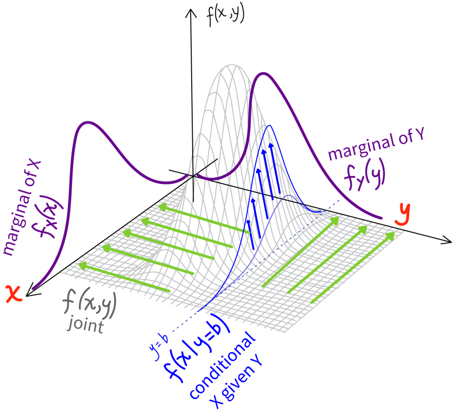
$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1.$$

- To get the marginal PDF of y_1, integrate over y_2.
- To get the marginal PDF of y_2, integrate over y_1.

Marginal PDF

Other Remarks: Suppose we have random variables X and Y.

- The marginal PDF of x is free of y.
  - Marginalization projects or flattens the 3D density to 2D onto the x-axis.
- The marginal PDF of y is free of x.
  - Marginalization projects or flattens the 3D density to 2D onto the y-axis.



Marginal PDF

Example 2:

Suppose (X, Y) takes values on the square $[0,1] \times [1,2]$ with joint PDF:

$$f(x,y) = \frac{8}{3}x^3y.$$

Find the marginal PDF of X and Y.

Solution:

- Computing the marginal PDF of X: Formula: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$f_X(x) = \int_1^2 \frac{8}{3}x^3ydy = \frac{8}{3}x^3 \int_1^2 ydy = \frac{8}{3}x^3\left(\frac{y^2}{2}\right)\Big|_1^2 = 4x^3.$$

- Computing the marginal PDF of Y: Formula: $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$

$$f_Y(y) = \int_0^1 \frac{8}{3}x^3ydx = \frac{8}{3}y \int_0^1 x^3dx = \frac{8}{3}y\left(\frac{x^4}{4}\right)\Big|_0^1 = \frac{2}{3}y.$$

☐

Marginal PDF

Example 3:

Suppose (X, Y) takes values on the unit square with joint PDF:

$$f(x,y) = \frac{3}{2}(x^2 + y^2).$$

What is $P(X < 0.5)$?

Solution:

- We first need to compute the marginal PDF of X: Formula: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$f_X(x) = \int_0^1 \frac{3}{2}(x^2 + y^2)dy = \frac{3}{2}\left(x^2 y + \frac{y^3}{3}\right)\Big|_0^1 = \frac{3}{2}x^2 + \frac{1}{2}.$$

- Computing $P(X < 0.5)$:

$$P(X < 0.5) = \int_{-\infty}^{0.5} f_X(x)dx = \int_0^{0.5} \frac{3}{2}x^2 + \frac{1}{2}dx = \left(\frac{x^3}{2} + \frac{x}{2}\right)\Big|_0^{0.5} = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}.$$

□

Marginal PDF

Example 4:

Suppose (X, Y) is a bivariate Gaussian random variable with PDF:

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{\{(x-\mu_X)^2+(y-\mu_Y)^2\}}{2\sigma^2}}.$$

What is the marginal distribution of X?

Solution:

- We first need to find the marginal PDF of X: Formula: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{\{(x-\mu_X)^2+(y-\mu_Y)^2\}}{2\sigma^2}} dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_X)^2}{2\sigma^2}} e^{-\frac{(y-\mu_Y)^2}{2\sigma^2}} dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_X)^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_Y)^2}{2\sigma^2}} dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_X)^2}{2\sigma^2}} (1).
\end{aligned}$$

Gaussian PDF integrates to 1.

- The marginal distribution of X is $\mathcal{N}(\mu_x, \sigma^2)$. ☐

Conditional Probability Distributions

Conditional Probability Distributions

- Recall our discussion on conditional probability:
  - As you obtain additional information, how should you update probabilities of events?
  - Formula:

$$\begin{aligned} P(A|B) &= \frac{\text{probability of events A and B both occurring}}{\text{probability of event B occurring}} \\ &= \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) > 0. \end{aligned}$$

- Same principle applies to distributions:
  - conditional PMF: $p(y_1|y_2) = \frac{p(y_1,y_2)}{p_2(y_2)}$
  - conditional PDF: $f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)}$
  - conditional CDF: $F(y_1|y_2) = P(Y_1 \leq y_1|Y_2 = y_2)$
- Conditional distributions are used to model scenarios where the values of a subset of the random variables are known and the rest are missing or unknown.

Conditional PMF

Definition: Conditional PMF for Discrete Random Variables

Let Y_1 and Y_2 be jointly discrete random variables with joint PMF $p(y_1, y_2)$ and marginal PMFs $p_1(y_1)$ and $p_2(y_2)$. The *conditional PMF* of Y_1 given $Y_2 = y_2$ is

$$p(y_1|y_2) = P(Y_1 = y_1|Y_2 = y_2) = \frac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_2 = y_2)} = \frac{p(y_1, y_2)}{p_2(y_2)},$$

provided that $p_2(y_2) > 0$.

- To compute the conditional PMF, we need:
  - joint PMF: $p(y_1, y_2)$
  - marginal PMF of the conditioning RV: $p_2(y_2)$

$$\text{conditional PMF} = \frac{\text{joint PMF}}{\text{marginal PMF of conditioning RV}}$$

Conditional PMF

Example 5:

Suppose X and Y have the following joint PMF:

		y		
	$X\backslash Y$	0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

Given that $X = 1$, determine the conditional PMF of Y.

Solution:

$$\begin{aligned} p(0|1) &= P(Y=0|X=1) \\ &= \frac{P(Y=0, X=1)}{P(X=1)} \\ &= \frac{0.08}{P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2)} \\ &= \frac{0.08}{0.08+0.20+0.06} = \frac{0.08}{0.34} = 0.2353. \end{aligned}$$

		y		
	$X\backslash Y$	0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

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Conditional PMF

Example 5:

Suppose X and Y have the following joint PMF:

			y	
	$X\backslash Y$	0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

Given that $X = 1$, determine the conditional PMF of Y.

Solution:

			y	
	$X\backslash Y$	0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

$$\begin{aligned} p(1|1) &= P(Y = 1|X = 1) \\ &= \frac{P(Y = 1, X = 1)}{P(X = 1)} \\ &= \frac{0.20}{P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2)} \\ &= \frac{0.20}{0.08 + 0.20 + 0.06} = \frac{0.20}{0.34} = 0.5882. \end{aligned}$$

(cont'd next slide...)

Conditional PMF

Example 5:

Suppose X and Y have the following joint PMF:

			y	
	$X\backslash Y$	0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

Given that $X = 1$, determine the conditional PMF of Y.

Solution:

$$\begin{aligned} p(2|1) &= P(Y = 2|X = 1) \\ &= \frac{P(Y = 2, X = 1)}{P(X = 1)} \\ &= \frac{0.06}{P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2)} \\ &= \frac{0.06}{0.08 + 0.20 + 0.06} = \frac{0.06}{0.34} = 0.1765. \end{aligned}$$

			y	
	$X\backslash Y$	0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

(cont'd next slide...)

Conditional PMF

Example 5:

Suppose X and Y have the following joint PMF:

			y	
	$X\backslash Y$	0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

Given that $X = 1$, determine the conditional PMF of Y.

Solution:

The conditional PMF of Y is

$$p(y|X=1) = \begin{cases} 0.2353, & \text{if } y = 0 \\ 0.5882, & \text{if } y = 1 \\ 0.1765, & \text{if } y = 2. \end{cases}$$

Conditional CDF

Definition: Conditional CDF for Continuous Random Variables

Let Y_1 and Y_2 be jointly continuous random variables with joint PDF $f(y_1, y_2)$. The *conditional CDF* of Y_1 given $Y_2 = y_2$ is

$$F(y_1|y_2) = P(Y_1 \leq y_1|Y_2 = y_2).$$

- Discrete Case:

$$F(y_1|y_2) = \frac{1}{p_2(y_2)} \sum_{t \leq y_1} p(t, y_2)$$

- Continuous Case:

$$F(y_1|y_2) = \frac{1}{f_2(y_2)} \int_{-\infty}^{y_1} f(t, y_2)dt$$

Conditional CDF

Example 6:

Suppose X and Y have the following joint PMF:

		y		
	$X\backslash Y$	0	1	2
x	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

What is $P(Y \leq 1 | X = 2)$?

Solution:

		y		
	$X\backslash Y$	0	1	2
x	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

$$
\begin{aligned}
P(Y \leq 1 | X = 2) &= P(Y = 0 | X = 2) + P(Y = 1 | X = 2) \\
&= \frac{P(Y = 0, X = 2)}{P(X = 2)} + \frac{P(Y = 1, X = 2)}{P(X = 2)} \\
&= \frac{0.06 + 0.14}{P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2)} \\
&= \frac{0.06 + 0.14}{0.06 + 0.14 + 0.30} = \frac{0.20}{0.50} = 0.4. \quad \square
\end{aligned}
$$

Conditional PDF

Definition: Conditional PDF for Continuous Random Variables

Let Y_1 and Y_2 be jointly continuous random variables with joint PDF $f(y_1, y_2)$ and marginal PDFs $f_1(y_1)$ and $f_2(y_2)$. The *conditional PDF* of Y_1 given $Y_2 = y_2$ is

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)},$$

provided that $f_2(y_2) > 0$.

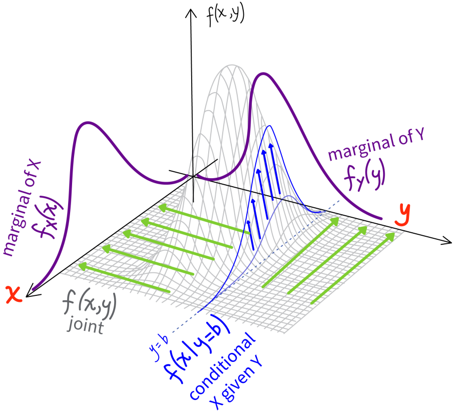
- To compute the conditional PDF, we need:
  - joint PDF: $f(y_1, y_2)$
  - marginal PDF of the conditioning RV: $f_2(y_2)$

$$\text{conditional PDF} = \frac{\text{joint PDF}}{\text{marginal PDF of conditioning RV}}$$

Conditional PDF

Other Remarks: Suppose we have random variables X and Y.

- The conditional density of X given $Y = b$ is the 2D image of the 3D density at $y = b$, divided by $f_Y(b)$.



Conditional PDF

Example 7:

Let X and Y have the following joint density:

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x, y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

- ⓐ Determine the conditional PDF of Y given X.

Solution:

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f_X(x)} \qquad \text{conditional PDF formula} \\ &= \frac{\frac{6}{5}(x+y^2)}{\int_0^1 f(x,y)dy} \qquad \frac{\text{given}}{\text{marginal PDF formula: } f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy} \\ &= \frac{\frac{6}{5}(x+y^2)}{\int_0^1 \frac{6}{5}(x+y^2)dy} = \frac{\frac{6}{5}(x+y^2)}{\frac{6}{5}\left(xy + \frac{y^3}{3}\right)\Big|_0^1} = \frac{x+y^2}{x+\frac{1}{3}}. \end{aligned}$$

□

Conditional PDF

Example 7:

Let X and Y have the following joint density:

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x, y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

ⓑ Find $P(Y \ge \frac{1}{2} | X = \frac{1}{3})$.

Solution: (to get the conditional prob., integrate the conditional PDF)

Use the conditional PDF from part a) and set $x = \frac{1}{3}$.

$$\begin{aligned} P\left(Y \ge \frac{1}{2}\Big|X = \frac{1}{3}\right) &= \int_{\frac{1}{2}}^{1} f(y|x=\frac{1}{3})dy \\ &= \int_{\frac{1}{2}}^{1} \frac{(\frac{1}{3}+y^2)}{(\frac{1}{3}+\frac{1}{3})}dy \quad \text{conditional PDF from part a) is } f(y|x) = \frac{x+y^2}{x+\frac{1}{3}} \\ &= \frac{1}{\frac{2}{3}} \int_{\frac{1}{2}}^{1} \frac{1}{3} + y^2 dy = \frac{3}{2}\left(\frac{1}{3}y + \frac{y^3}{3}\right)\Big|_{\frac{1}{2}}^{1} \\ &= \frac{3}{2}\left\{\left(\frac{1}{3}+\frac{1}{3}\right) - \left(\frac{1}{6}+\frac{1}{24}\right)\right\} = \frac{11}{16}. \quad \square \end{aligned}$$

Independent Random Variables

Independent Random Variables

- Recall our discussion on independence:
  - Two events are said to be independent if knowing one occurs does not change the probability of the other occurring.
  - Any one of the following should hold:

$$\begin{aligned} P(A|B) &= P(A) \\ P(B|A) &= P(B) \\ P(A \cap B) &= P(A)P(B) \end{aligned}$$

  Otherwise, the events are said to be *dependent*.
- Same principle applies to random variables

Independent Random Variables

Definition: Checking for Independence via CDF

Suppose Y_1 have CDF $F_1(y_1)$, Y_2 have CDF $F_2(y_2)$, and Y_1 and Y_2 have joint CDF $F(y_1, y_2)$. Then Y_1 and Y_2 are said to be *independent* if and only if

$$F(y_1, y_2) = F_1(y_1)F_2(y_2),$$

for every pair of real numbers (y_1, y_2).

If Y_1 and Y_2 are not independent, they are said to be *dependent*.

- To check for independence, we need:
    - joint CDF: $F(y_1, y_2)$
    - marginal CDFs: $F_1(y_1)$, $F_2(y_2)$

Independent Random Variables

Theorem: Checking for Independence via PMF (Discrete)

Let Y_1 and Y_2 be jointly discrete random variables with joint PMF $p(y_1, y_2)$ and marginal PMFs $p_1(y_1)$ and $p_2(y_2)$. Then Y_1 and Y_2 are independent if and only if

$$p(y_1, y_2) = p_1(y_1)p_2(y_2),$$

for every pair of real numbers (y_1, y_2).

Independent Random Variables

Example 8: (Recall Example 1 in Lecture 15)

Roll two dice. Let X and Y be the value on the first and second die, respectively. Are X and Y independent?

Solution:

- From Ex. 1 in Lec 15, we have the joint PMF of X and Y.

		y						
	$X\backslash Y$	1	2	3	4	5	6	$p_X(x)$
x	1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p_Y(y)$		1/6	1/6	1/6	1/6	1/6	1/6	

- Since every marginal probability is 1/6 and every cell contains 1/36, X and Y are independent.

Independent Random Variables

Example 9: (Recall Example 2 in Lecture 15)

Roll two dice. Let X be the value on the first die and let T be the total on both dice. Are X and Y independent?

Solution:

- From Ex. 2 in Lec 15, we have the joint PMF of X and T.

		t											
	$X\backslash T$	2	3	4	5	6	7	8	9	10	11	12	$p_X(x)$
x	1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0	1/6
	2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	1/6
	3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	1/6
	4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	1/6
	5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	1/6
	6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	$p_Y(y)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	

- Since most of the cell probabilities are not the product of the marginal probabilities, X and Y cannot be independent.

Independent Random Variables

Theorem: Checking for Independence via PDF (Continuous)

Let Y_1 and Y_2 be jointly continuous random variables with joint PDF $f(y_1, y_2)$ and marginal PDFs $f_1(y_1)$ and $f_2(y_2)$. Then Y_1 and Y_2 are independent if and only if

$$f(y_1, y_2) = f_1(y_1)f_2(y_2),$$

for every pair of real numbers (y_1, y_2).

Independent Random Variables

Example 10:

Consider X and Y with the joint density:

$$f(x,y) = \begin{cases} x+y, & 0 < x, y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Are X and Y independent?

Solution:

- Solving for the marginal PDF of X:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \quad \text{marginal PDF formula} \\ &= \int_0^1 x+y\ dy \quad \text{given} \\ &= \left(xy + \frac{y^2}{2}\right)\Big|_0^1 = x + \frac{1}{2}, \quad 0 < x < 1. \end{aligned}$$

(cont'd next slide...)

Independent Random Variables

Example 10:

Consider X and Y with the joint density:

$$f(x,y) = \begin{cases} x+y, & 0 < x,y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Are X and Y independent?

Solution:

- Solving for the marginal PDF of Y:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \quad \text{marginal PDF formula} \\ &= \int_0^1 x+y \; dx \quad \text{given} \\ &= \left(\frac{x^2}{2} + xy\right)\Big|_0^1 = \frac{1}{2} + y, \quad 0 < y < 1. \end{aligned}$$

(cont'd next slide...)

Independent Random Variables

Example 10:

Consider X and Y with the joint density:

$$f(x,y) = \begin{cases} x+y, & 0 < x, y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Are X and Y independent?

Solution:

- Checking if $f(x,y) = f_X(x)f_Y(y)$:

$$\begin{aligned} f_X(x)f_Y(y) &= \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \\ &= xy+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{4} \\ &\neq f(x,y). \end{aligned}$$

- Therefore, X and Y are NOT independent.

Independent Random Variables: Via Conditioning

Independence means "conditioning does not change the distribution"...

Recall formula for conditional PDF:

$$\begin{aligned} f(y_1|y_2) &= \frac{f(y_1, y_2)}{f_2(y_2)} \\ \Leftrightarrow f(y_1, y_2) &= f(y_1|y_2) f_2(y_2). \end{aligned}$$

Recall the independence property via PDFs:

$$\begin{aligned} f(y_1, y_2) &= f_1(y_1) f_2(y_2) \\ \Leftrightarrow f(y_1|y_2) f_2(y_2) &= f_1(y_1) f_2(y_2) \\ \Leftrightarrow f(y_1|y_2) &= f_1(y_1). \end{aligned}$$

$$\text{independence} \Longleftrightarrow \text{conditional PDF} = \text{marginal PDF}$$

Same principle applies to disrete RVs...

Independent Random Variables

Theorem: Useful Theorem for Independence

Let Y_1 and Y_2 have a joint PDF $f(y_1, y_2)$ that is positive if and only if $a \leq y_1 \leq b$ and $c \leq y_2 \leq d$, for constants a, b, c, and d, and $f(y_1, y_2) = 0$ otherwise. Then Y_1 and Y_2 are independent if and only if

$$f(y_1, y_2) = g(y_1)h(y_2),$$

where $g(y_1)$ is a nonnegative function of y_1 alone and $h(y_2)$ is a nonnegative function of y_2 alone.

- Independence means you can factor the joint PDF as the product of a function of y_1 and a function of y_2.
- This tells us that we do not actually need to derive the marginal densities.
- $g(y_1)$ and $h(y_2)$ need not, themselves, be density functions.

Independent Random Variables

Example 11:

Suppose X and Y has joint PDF $f(x, y) = 96x^2y^3$.
Are X and Y independent?

Solution:

- Try to find functions $g(x)$ and $h(y)$, where
  - $g(x)$ is solely a function of x: $g(x) = 24x^2$
  - $h(y)$ is solely a function of y: $h(y) = 4y^3$,

  such that their product is $f(x, y) = 96x^2y^3$.
- By the previous theorem, X and Y are independent.

Questions?

Homework Exercises: 5.21, 5.23, 5.27, 5.35, 5.41

Solutions will be discussed this Friday by the TA.