STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 18: Functions of Random Variables (Univariate)

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April 8, 2024

Outline

1. Midterm 2 Solutions
2. Previously...
3. Functions of Random Variables (Univariate)
   - The CDF Method
   - The Jacobian Transformation Method
   - The MGF Method

Midterm 2 Solutions

Problem 1

Suppose X and Y are independent Gaussian random variables. That is, $X \sim \mathcal{N}(1, 4)$ and $Y \sim \mathcal{N}(0, 7)$.

a. Find $\text{Cov}(X, Y)$.

Solution:

Since X and Y are independent, $\text{Cov}(X, Y) = 0$. □

Problem 1

Suppose X and Y are independent Gaussian random variables. That is, $X \sim \mathcal{N}(1, 4)$ and $Y \sim \mathcal{N}(0, 7)$.

(b) Find $E(X^2 Y^2)$.

Solution:

- Since X and Y are independent, we can split the expected value as follows:

$$E(X^2 Y^2) = E(X^2)E(Y^2).$$

- Solving for $E(X^2)$, we have

$$\begin{aligned} E(X^2) &= V(X) + \{E(X)\}^2 && \text{def'n of variance} \\ &= 4 + 1^2 && X \sim \mathcal{N}(1,4) \\ &= 5. \end{aligned}$$

- Solving for $E(Y^2)$, we have

$$\begin{aligned} E(Y^2) &= V(Y) + \{E(Y)\}^2 && \text{def'n of variance} \\ &= 7 + 0^2 && Y \sim \mathcal{N}(0,7) \\ &= 7. \end{aligned}$$

- Therefore, $E(X^2 Y^2) = E(X^2)E(Y^2) = (5)(7) = 35.$ □

Problem 1

Suppose X and Y are independent Gaussian random variables. That is, $X \sim \mathcal{N}(1, 4)$ and $Y \sim \mathcal{N}(0, 7)$.

(c) Find $E(3X - 2Y)$.

Solution:

By the linearity of expectation, we have

$$
\begin{aligned}
E(3X - 2Y) &= 3E(X) - 2E(Y) \\
&= 3(1) - 2(0) \quad \text{$X \sim \mathcal{N}(1,4)$ and $Y \sim \mathcal{N}(0,7)$} \\
&= 3.
\end{aligned}
$$

□

Problem 1

Suppose X and Y are independent Gaussian random variables. That is, $X \sim \mathcal{N}(1, 4)$ and $Y \sim \mathcal{N}(0, 7)$.

(d) Find $V(3X - 2Y)$.

Solution:

$$
\begin{aligned}
V(3X - 2Y) &= V(3X) + V(-2Y) \\
&\quad \text{Variance of the sum of independent RVs: } V(X+Y) = V(X) + V(Y) \\
&= 3^2 V(X) + (-2)^2 V(Y) \quad \text{Variance of a linear transform: } V(aX+b) = a^2 V(X) \\
&= 9(4) + 4(7) \quad X \sim \mathcal{N}(1,4) \text{ and } Y \sim \mathcal{N}(0,7) \\
&= 64.
\end{aligned}
$$

□

Problem 1

Suppose X and Y are independent Gaussian random variables. That is, $X \sim \mathcal{N}(1, 4)$ and $Y \sim \mathcal{N}(0, 7)$.

- (e) Find $P(-3 \leq 3X - 2Y \leq 5)$. Hint: Sum of 2 Gaussian RVs is a Gaussian RV.

Solution:

- Let $W = 3X - 2Y$.
- From part c) and d), we know that $W \sim \mathcal{N}(\mu = 3, \sigma^2 = 64)$.

$$
\begin{aligned}
P(-3 \leq W \leq 5) &= P\left(\frac{-3-\mu}{\sigma} \leq \frac{W-\mu}{\sigma} \leq \frac{5-\mu}{\sigma}\right) && \text{standardization} \\
&= P\left(\frac{-3-3}{\sqrt{64}} \leq \frac{W-3}{\sqrt{64}} \leq \frac{5-3}{\sqrt{64}}\right) \\
&= P\left(-\frac{6}{8} \leq Z \leq \frac{2}{8}\right) \\
&= \Phi\left(\frac{1}{4}\right) - \Phi\left(-\frac{3}{4}\right) && \text{probability} = \text{area under the standard normal curve} \\
&= 0.59871 - 0.22663 && \text{Z-table values} \\
&= 0.3721. \quad \square
\end{aligned}
$$

Problem 2

Consider a random variable X with the PDF

$$f(x) = A + Bx^2, \quad 0 \leq x \leq 2.$$

If $E(X) = 1/2$, find A and B.

Solution:

- There are two unknowns, A and B. We will need two linear equations to find their values.
- Since $f(x)$ is a valid density, it must integrate to 1.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= \int_0^2 A + Bx^2 dx \\ &= Ax + B\frac{x^3}{3}\Big|_0^2 \\ &= 2A + \frac{8}{3}B. \end{aligned}$$

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Problem 2

Consider a random variable X with the PDF

$$f(x) = A + Bx^2, \quad 0 \leq x \leq 2.$$

If $E(X) = 1/2$, find A and B.
Solution:

- Also, we need $E(X) = 1/2$. This means

$$\begin{aligned} \frac{1}{2} &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{0}^{2} x(A + Bx^2)dx \\ &= A\frac{x^2}{2} + B\frac{x^4}{4}\Big|_0^2 \\ &= 2A + 4B. \end{aligned}$$

(cont'd next slide...)

Problem 2

Consider a random variable X with the PDF

$$f(x) = A + Bx^2, \quad 0 \leq x \leq 2.$$

If $E(X) = 1/2$, find A and B.

Solution:

- Solving the following system of linear equations, we have

$$\begin{cases} 2A + \frac{8}{3}B = 1 \\ 2A + 4B = \frac{1}{2} \end{cases}$$

$$\Rightarrow \frac{4}{3}B = -\frac{1}{2} \quad \text{subtracting the 1st eqn from the 2nd}$$

$$\Rightarrow B = -\frac{3}{8}$$

$$\Rightarrow 2A + \frac{8}{3}\left(-\frac{3}{8}\right) = 1 \quad \text{substituting the value of } B \text{ to the 1st eqn}$$

$$\Rightarrow A = 1.$$

Thus, we have the following PDF:

$$f(x) = 1 - \frac{3}{8}x^2, \quad 0 \leq x \leq 2. \quad \square$$

Problem 3

Suppose that the completion time in hours T for the STAT 3375Q final exam follows a distribution with density

$$f(t) = \frac{2}{27}(t^2 + t), \quad 0 \le t \le 3.$$

What is the probability that a randomly chosen student finishes the exam during the first 30 minutes.

Solution:

$$\begin{aligned} P\left(T \le \frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \frac{2}{27}(t^2 + t)dt \quad \text{probability = area under density curve} \\ &= \frac{2}{27}\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Big|_0^{\frac{1}{2}} = \frac{2}{27}\left\{\frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{2}\right)^2\right\} \\ &= \frac{2}{27}\left(\frac{1}{24} + \frac{1}{8}\right) = \frac{2}{27}\left(\frac{4}{24}\right) \\ &= \frac{1}{81}. \quad \square \end{aligned}$$

Problem 4

Given that X has MGF

$$m(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t},$$

what is the probability that X is even.

Solution:

Matching the MGF above to the MGF formula $m(t) = E(e^{tX}) = \sum_y e^{tx}p(x)$, we know that the given MGF corresponds to a discrete random variable with PMF:

$$p(x) = \begin{cases} \frac{1}{6}, & \text{if } x = -2, \\ \frac{1}{3}, & \text{if } x = -1, \\ \frac{1}{4}, & \text{if } x = 1, \\ \frac{1}{4}, & \text{if } x = 2. \end{cases}$$

Therefore,

$$P(X \text{ is even}) = P(X = -2) + P(X = 2) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}. \quad \square$$

Problem 5

Suppose X and Y are continuous random variables with joint PDF

$$f(x, y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1; 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the marginal PDF of X, $f(x)$, and Y, $f(y)$.

Solution:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\ &= \int_0^1 4xy dy \\ &= 4x \frac{y^2}{2}\bigg|_0^1 \\ &= 2x, \quad 0 \le x \le 1. \end{aligned}$$

Similarly, $f(y) = 2y$, $0 \le y \le 1$. □

Problem 5

Suppose X and Y are continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1; 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

b Find the conditional PDF of Y given X, $f(y|x)$.

Solution:

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f(x)} \\ &= \frac{4xy}{2x} \\ &= 2y, \quad 0 \le y \le 1. \end{aligned}$$

□

Problem 5

Suppose X and Y are continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1; 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

c. Find $P(Y \le 3/4 | X = 1/2)$.

Solution:

$$\begin{aligned} P(Y \le 3/4|X = 1/2) &= \int_0^{3/4} f(y|x = 1/2)dy \\ &= \int_0^{3/4} 2ydy \quad \text{Using the conditional PDF in part b)} \\ &= y^2\Big|_0^{3/4} = \frac{9}{16}. \end{aligned}$$

□

Problem 5

Suppose X and Y are continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 4xy, & \text{if } 0 \leq x \leq 1; 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(d) Find $E(Y|X=x)$.

Solution:

$$\begin{aligned} E(Y|X=x) &= \int_{-\infty}^{\infty} y f(y|x) dy \quad \text{def'n of conditional expectation} \\ &= \int_0^1 y(2y) dy = \frac{2y^3}{3}\bigg|_0^1 = \frac{2}{3}. \end{aligned}$$

□

Problem 6

Let X be a random variable with MGF

$$m(t) = \begin{cases} \frac{e^t - e^{-t}}{2t}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0. \end{cases}$$

(a) Give the distribution of X.

Solution:

Matching the MGF above with known MGF formulas, we know that $X \sim U(-1, 1)$, where $\theta_1 = -1$ and $\theta_2 = 1$. □

Problem 6

Let X be a random variable with MGF

$$m(t) = \begin{cases} \frac{e^t - e^{-t}}{2t}, & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

(b) Compute $E(X)$ and $V(X)$.

Solution:

Using the mean and variance formula of a uniform RV, we have

$$\begin{aligned} E(X) &= \frac{\theta_1 + \theta_2}{2} = \frac{-1+1}{2} = 0 \\ V(X) &= \frac{(\theta_2 - \theta_1)^2}{12} = \frac{\{1-(-1)\}^2}{12} = \frac{4}{12} = \frac{1}{3}. \end{aligned}$$

□

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad E(X^2) = 3, \quad E(XY) = -4, \quad E(Y) = 2, \quad V(Y) = 25.$$

(a) Find $E(2X + Y)$.

Solution:

$$\begin{aligned} E(2X+Y) &= 2E(X) + E(Y) && \text{linearity of expectation} \\ &= 2(1) + 2 && \text{given} \\ &= 4. \end{aligned}$$

□

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad E(X^2) = 3, \quad E(XY) = -4, \quad E(Y) = 2, \quad V(Y) = 25.$$

(b) Find $E\{X(2X + Y)\}$.

Solution:

$$\begin{aligned} E\{X(2X+Y)\} &= E(2X^2 + XY) \\ &= 2E(X^2) + E(XY) \quad \text{linearity of expectation} \\ &= 2(3) + (-4) \quad \text{given} \\ &= 2. \end{aligned}$$

□

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad E(X^2) = 3, \quad E(XY) = -4, \quad E(Y) = 2, \quad V(Y) = 25.$$

c. Find $\text{Cov}(X, 2X + Y)$.

Solution:

$$\begin{aligned}
\text{Cov}(X, 2X + Y) &= E\{X(2X + Y)\} - E(X)E(2X + Y) \\
&\quad \text{def'n of covariance: } \text{Cov}(X, Y) = E(XY) - E(X)E(Y) \\
&= 2 - (1)(4) \quad \text{answers from part a) and b)} \\
&= -2.
\end{aligned}$$

□

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad E(X^2) = 3, \quad E(XY) = -4, \quad E(Y) = 2, \quad V(Y) = 25.$$

(d) Find $V(2X + Y)$.

Solution:

$$V(2X+Y) = 2^2 V(X) + V(Y) + 2\text{Cov}(2X, Y)$$

Variance of the sum: $V(X+Y) = V(X) + V(Y) + 2\text{Cov}(X,Y)$

$$= 4V(X) + V(Y) + 2(2)\text{Cov}(X, Y)$$

Covariance of linear transform: $\text{Cov}(aX+b, cY+d) = ac\text{Cov}(X,Y)$. Here $a = 2, c = 1$.

$$= 4[E(X^2) - \{E(X)\}^2] + 25 + 4\{E(XY) - E(X)E(Y)\}$$

def'n of variance and covariance

$$= 4(3 - 1^2) + 25 + 4(-4 - (1)(2))$$

$$= 9.$$

Problem 7

Let X and Y be random variables such that

$$E(X) = 1, \quad E(X^2) = 3, \quad E(XY) = -4, \quad E(Y) = 2, \quad V(Y) = 25.$$

e. Find $\text{Corr}(X, 2X + Y)$.

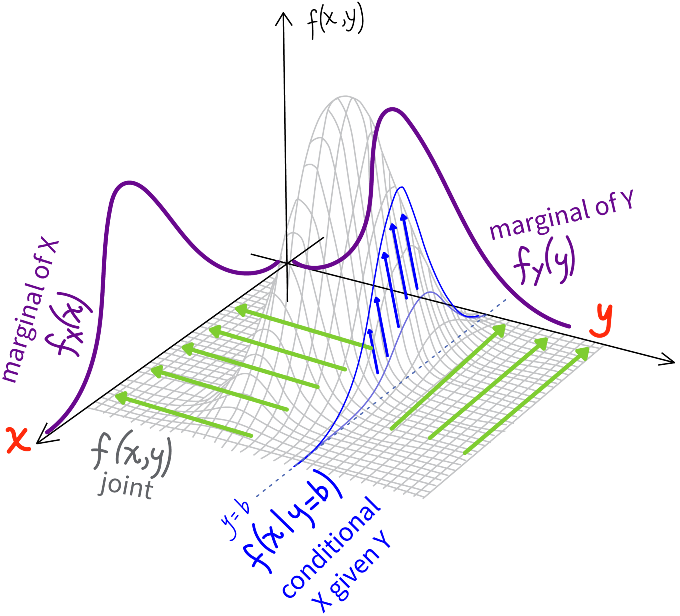
Solution:

$$\begin{aligned} \text{Corr}(X, 2X+Y) &= \frac{\text{Cov}(X, 2X+Y)}{\sqrt{V(X)V(2X+Y)}} && \text{def'n of correlation} \\ &= \frac{-2}{\sqrt{(2)(9)}} && \text{answers from part c) and d)} \\ &= \frac{-2}{\sqrt{18}} = -0.47. \end{aligned}$$

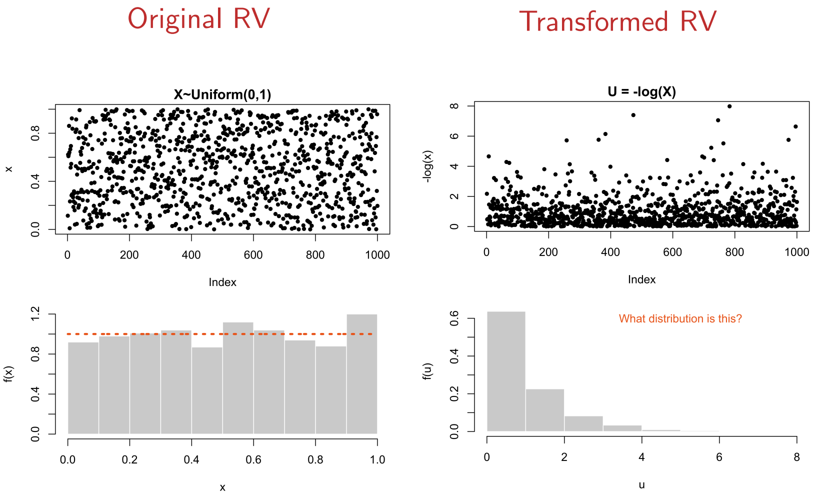
□

Previously...

Univariate & Multivariate



Next Few Topics: $X \to h(X)$



Functions of Random Variables (Univariate)

Functions of Random Variables (Univariate)

- Suppose we have a random variable X with PDF $f_X(x)$.
- Define a new random variable $U = h(X)$, where h is a (one-to-one) monotone function.
  - $h(x) = e^x$
  - $h(x) = \ln(x)$
  - $h(x) = \sqrt{x}$
  - $h(x) = x^2$
- What is the PDF of U?
  - CDF Method
  - Jacobian Method (PDF-to-PDF Method or change of variable)
  - MGF Method

The CDF Method

The CDF Method

Theorem: The CDF Method

Let X be a random variable with CDF $F_X(x)$. Define $U = h(X)$ where h is a monotone function. Let the domain of X and codomain of U be $\mathcal{X} = \{x : f_X(x) > 0\}$ and $\mathcal{U} = \{u : u = h(x) \text{ for some } x \in \mathcal{X}\}$, respectively.

1. If h is an increasing function on $\mathcal{X}$, the CDF of U is given by

$$F_U(u) = F_X(h^{-1}(u)), \quad \text{for all } u \in \mathcal{U}.$$

2. If h is a decreasing function on $\mathcal{X}$, the CDF of U is given by

$$F_U(u) = 1 - F_X(h^{-1}(u)), \quad \text{for all } u \in \mathcal{U}.$$

The density of U, $f_U(u)$ can be obtained by differentiation as follows:

$$f_U(u) \quad = \quad \frac{d}{du} F_U(u).$$

The CDF Method

Proof:

$$
\begin{aligned}
F_U(u) &= P(U \le u) && \text{CDF def'n} \\
&= P\{h(X) \le u\} && U = h(X).
\end{aligned}
$$

If h is an increasing function:

$$
F_U(u) = P[h^{-1}\{h(X)\} \le h^{-1}(u)] \quad \text{applying inverse transformation}
$$

does not change the inequality since h^{-1} is also increasing.

This means that if $a < b$, then $h^{-1}(a) < h^{-1}(b)$.

$$
\begin{aligned}
&= P\{X \le h^{-1}(u)\} && \text{Def'n of inverse: f \& g are inverses iff } f(g(x)) = x. \\
&= F_X\{h^{-1}(u)\}. && \text{CDF def'n} \quad \square
\end{aligned}
$$

The CDF Method

Proof:

$$\begin{aligned} F_U(u) &= P(U \leq u) \quad \text{CDF def'n} \\ &= P\{h(X) \leq u\} \quad U = h(X). \end{aligned}$$

If h is a decreasing function:

$$F_U(u) = P[h^{-1}\{h(X)\} > h^{-1}(u)] \quad \text{applying inverse transformation}$$

NEED TO CHANGE the inequality since h^{-1} is also decreasing.

This means that if $a < b$, then $h^{-1}(a) > h^{-1}(b)$.

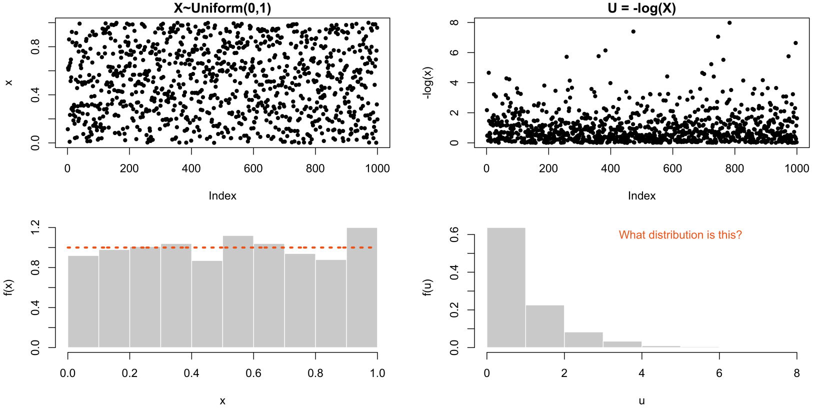
$$\begin{aligned} &= P\{X > h^{-1}(u)\} \quad \text{Def'n of inverse: f \& g are inverses iff } f(g(x)) = x. \\ &= 1 - P\{X \leq h^{-1}(u)\} \quad \text{complement} \\ &= 1 - F_X\{h^{-1}(u)\}. \quad \text{CDF def'n} \quad \square \end{aligned}$$

The CDF Method

Example 1:

Find the CDF of $U = h(X) = -\log X$, $X \sim \mathcal{U}(0,1)$.

Visualizing the problem...



The CDF Method

Example 1:

Find the CDF of $U = h(X) = -\log X$, $X \sim \mathcal{U}(0,1)$.

Solution:

- Domain of X: $x \in (0,1)$
- Codomain of U: $u = -\log(x) \Rightarrow u \in (0, \infty)$.
- CDF of a uniform RV X over $[0,1]$: $F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$

$$
\begin{aligned}
F_U(u) &= P(U \leq u) && \text{CDF def'n} \\
&= P(-\log X \leq u) && \text{given} \\
&= P(\log X > -u) && \text{multiply } -1 \text{ to both sides} \\
&= P(X > e^{-u}) && \text{take exponential of both sides} \\
&= 1 - P(X \leq e^{-u}) && \text{complement} \\
&= 1 - F_X(e^{-u}) && \text{CDF def'n} \\
&= 1 - e^{-u}. && \text{CDF of } X\text{, 2nd case since } 0 \leq e^{-u} \leq 1 \quad \square
\end{aligned}
$$

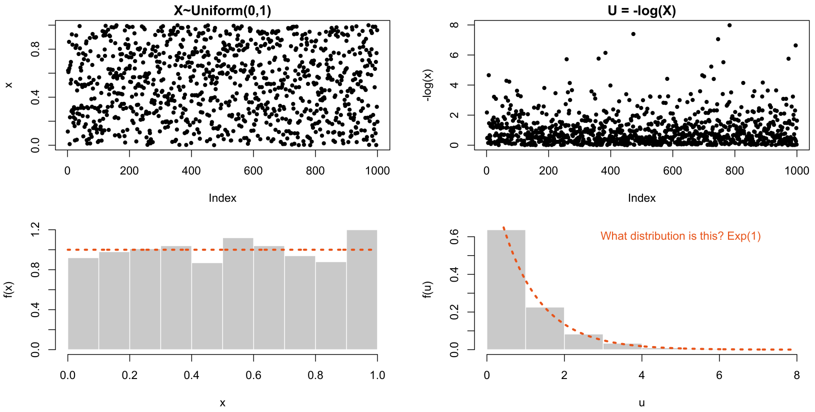
Note that this is the CDF of an exponential RV with $\beta = 1$.

The CDF Method

Example 1:

Find the CDF of $U = h(X) = -\log X$, $X \sim \mathcal{U}(0,1)$.

Visualizing the problem...



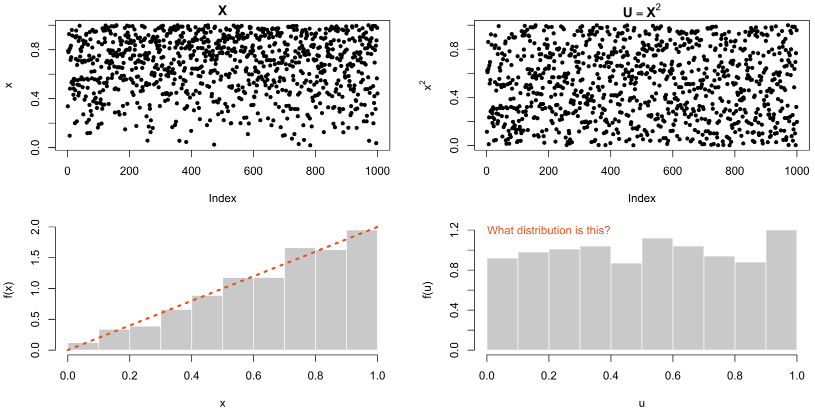
The CDF Method

Example 2:

Let X have the PDF $f_X(x) = 2x$, $0 \leq x \leq 1$.

Let $U = X^2$. Find the PDF of U.

Visualizing the problem...



The CDF Method

Example 2:

Let X have the PDF $f_X(x) = 2x$, $0 \leq x \leq 1$.
Let $U = X^2$. Find the PDF of U.

Solution:

- Domain of X: $x \in (0, 1)$
- Codomain of U: $u = x^2 \Rightarrow u \in (0, 1)$.
- CDF of X: $F_X(x) = \int_0^x 2tdt = \begin{cases} 0, & \text{if } x < 0, \\ x^2, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$

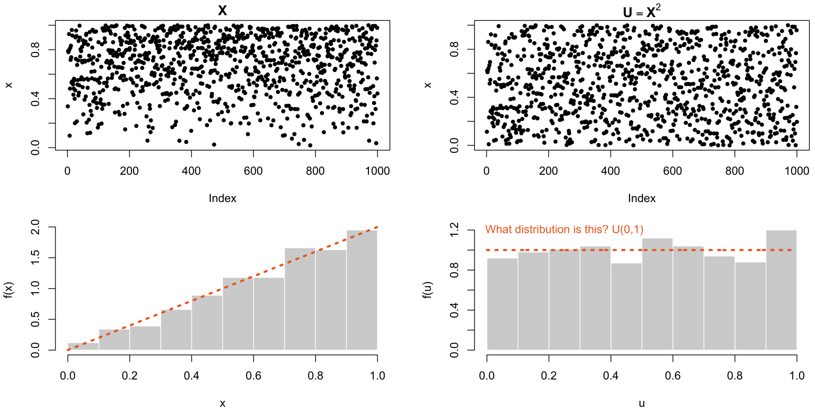
$$\begin{aligned} F_U(u) &= P(U \leq u) && \text{CDF def'n} \\ &= P(X^2 \leq u) && \text{given} \\ &= P(X \leq \sqrt{u}) && \text{isolate } X \\ &= u. && \text{CDF of } X\text{, 2nd case since } 0 \leq \sqrt{u} \leq 1 \\ f_U(u) &= \frac{d}{du} F_U(u) \\ &= 1. \quad \square \end{aligned}$$

Note that this is the PDF of a $U(0, 1)$ RV.

Example 2:

Let X have the PDF $f_X(x) = 2x$, $0 \leq x \leq 1$.
Let $U = X^2$. Find the PDF of U.

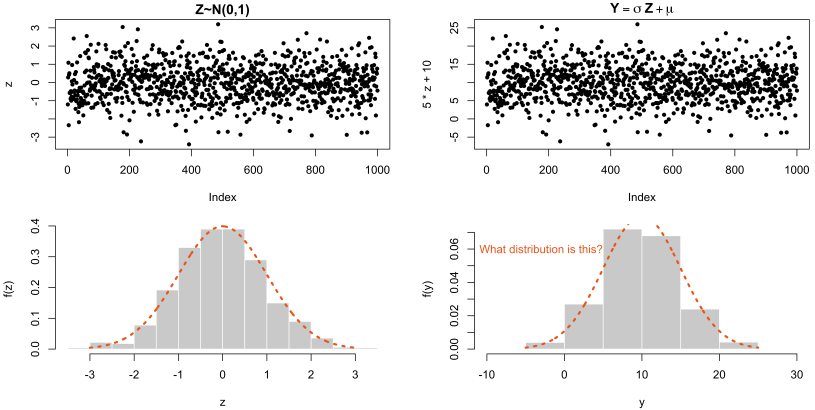
Visualizing the problem...



Example 3:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty \leq z \leq \infty$.
Let $Y = \sigma Z + \mu$. Find the PDF of Y.

Visualizing the problem...



The CDF Method

Example 3:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = \sigma Z + \mu$. Find the PDF of Y.

Solution:

- Domain of Z: $(-\infty, \infty)$
- Codomain of Y: $(-\infty, \infty)$

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) && \text{CDF def'n} \\
&= P(\sigma Z + \mu \leq y) && \text{transformation} \\
&= P\left(Z \leq \frac{y-\mu}{\sigma}\right) && \text{isolate the original RV} \\
&= \Phi\left(\frac{y-\mu}{\sigma}\right). && \text{CDF of the original RV: Standard Normal CDF } \Phi(z) \\
f_Y(y) &= \frac{d}{dy}\{F_Y(y)\} = \frac{d}{dy}\left\{\Phi\left(\frac{y-\mu}{\sigma}\right)\right\} \\
&= \frac{1}{\sigma}\left\{\phi\left(\frac{y-\mu}{\sigma}\right)\right\} && \text{derivative of CDF } \Phi(z) \text{ is PDF } \phi(z)\text{; chain rule} \\
&= \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}. \quad \square
\end{aligned}
$$

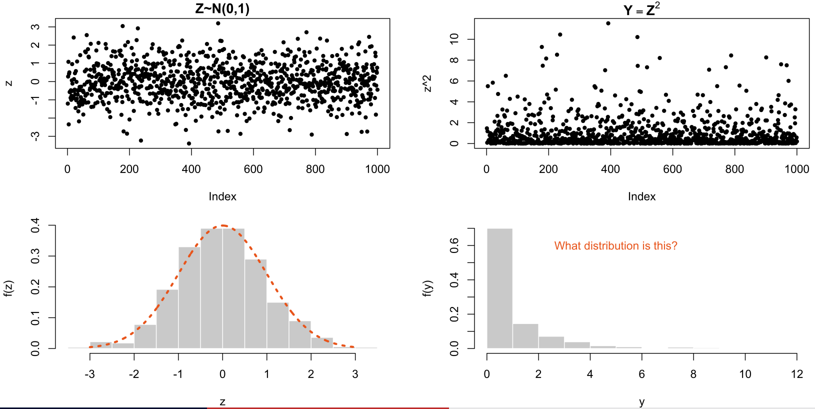
This is the Gaussian PDF.

The CDF Method

Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty \leq z \leq \infty.$
Let $Y = Z^2$. Find the PDF of Y.

Visualizing the problem...



Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.

Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Domain of Z: $(-\infty, \infty)$
- Codomain of Y: $(0, \infty)$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) && \text{CDF def'n} \\ &= P(Z^2 \leq y) && \text{transformation} \\ &= P\left(-\sqrt{y} \leq Z \leq \sqrt{y}\right) && \text{isolate the original RV} \\ &= P\left(Z \leq \sqrt{y}\right) - P\left(Z \leq -\sqrt{y}\right) \\ &= \Phi\left(\sqrt{y}\right) - \Phi\left(-\sqrt{y}\right). && \text{CDF of the original RV: Standard Normal CDF } \Phi(z) \end{aligned}$$

(cont'd next slide...)

The CDF Method

Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Domain of Z: $(-\infty, \infty)$
- Codomain of Y: $(0, \infty)$

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy}\{F_Y(y)\} = \frac{d}{dy}\{\Phi(\sqrt{y})\} - \frac{d}{dy}\{\Phi(-\sqrt{y})\} \\
&= \frac{1}{2} y^{-1/2}\{\phi(\sqrt{y})\} - \left[-\frac{1}{2} y^{-1/2}\{\phi(-\sqrt{y})\}\right]
\end{aligned}
$$

derivative of CDF $\Phi(z)$ is PDF $\phi(z)$; chain rule

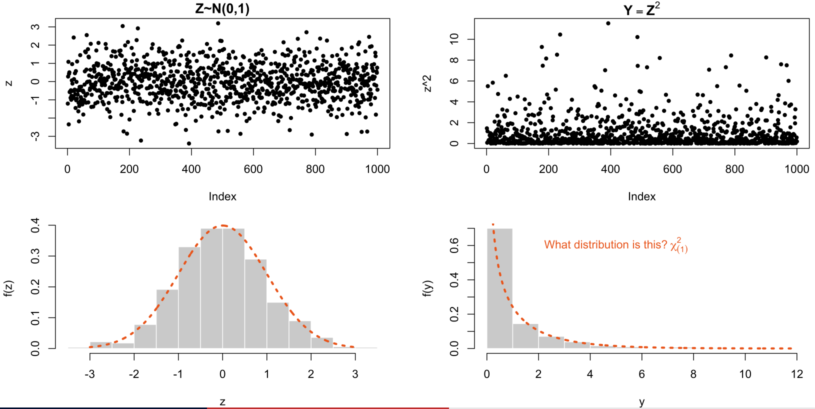
$$
\begin{aligned}
&= \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \\
&= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}. \quad \square
\end{aligned}
$$

This is the $\chi^2(1)$ PDF.

Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Visualizing the problem...



The Jacobian Transformation Method

The Jacobian Transformation Method

Theorem: The PDF-to-PDF Method (monotone)

Let X be a random variable with PDF $f_X(x)$. Define $U = h(X)$ where h is a differentiable *monotone* function for all values where $f_X(x) > 0$, such that the equation $u = h(x)$ can be solved for x to yield $x = h^{-1}(u)$. Then, the PDF of U has the form:

$$f_U(u) = f_X\{h^{-1}(u)\}\left|\frac{dh^{-1}(u)}{du}\right|,$$

where $|\cdot|$ is the absolute value function.

- $\frac{dh^{-1}(u)}{du}$ is called the Jacobian of the transformation h and accounts for the stretching of the interval caused by the transformation.
- The theory behind this method is more difficult than the CDF method but it is often easier to compute in practice.
- Advantage: direct computation of the PDF of U without the middle step of finding the CDF of U.

The Jacobian Transformation Method

Theorem: The PDF-to-PDF Method (non-monotone)

Let X be a random variable with PDF $f_X(x)$. Define $U = h(X)$ where h is a differentiable *non-monotone* function. The PDF of U has the form:

$$f_U(u) = \sum_{k=1}^{n(y)} f_X\{h_k^{-1}(u)\} \left| \frac{dh_k^{-1}(u)}{du} \right|,$$

where $n(y)$ is the number of invertible intervals, $h_k^{-1}(u)$ is the inverse transformation on the interval k, and $|\cdot|$ is the absolute value function.

- non-monotone transformations can be dealt with by splitting the transformation into intervals which are locally monotone...

Monotonic vs. Non-Monotonic Transformations

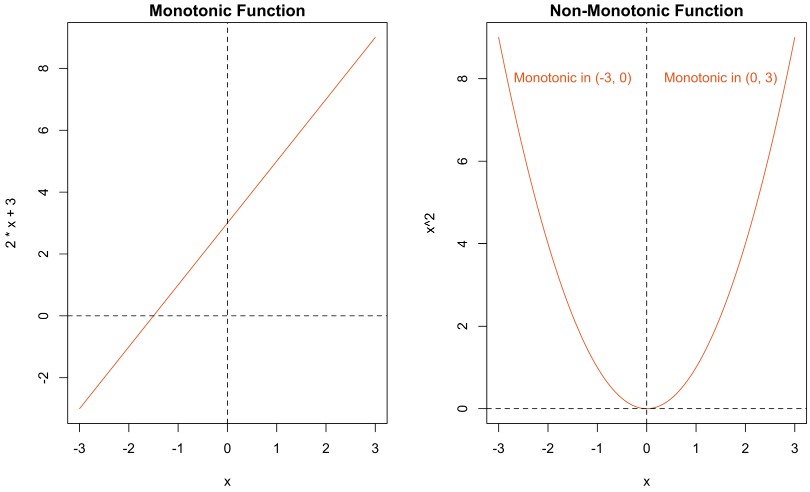
Definition: Monotonic Function

A *monotonic* function is a function whose first derivative does not change signs. Thus, it is always decreasing or always increasing, or always constant, but not more than one of these.

Definition: Non-Monotonic Function

A *non-monotonic* function is a function whose first derivative changes signs. Thus, it is increasing or decreasing for some time and shows opposite behavior at a different location.

Monotonic vs. Non-Monotonic Transformations



The Jacobian Transformation Method: How it Works

- We need the following to obtain the new PDF:
  - original PDF: $f_X(x)$
  - transformation function: $u = h(x)$
  - inverse of the transformation: $x = h^{-1}(u)$
  - Jacobian: $\frac{dh^{-1}(u)}{du}$

Re-doing Example 2:

Let X have the PDF $f_X(x) = 2x$, $0 \leq x \leq 1$.
Let $U = X^2$. Find the PDF of U.

Solution:

- Domain of X: $x \in (0, 1)$ given
- Codomain of U: $u \in (0, 1)$
- Transformation: $u = h(x) = x^2$ given
- Deriving the inverse of the transformation, $h^{-1}(u)$:
  (write x in terms of u)

$$\begin{aligned} u &= x^2 \\ \sqrt{u} &= x \end{aligned}$$

only take the positive root since $x \in (0, 1)$

Therefore, the inverse function is $h^{-1}(u) = \sqrt{u}$

- Jacobian: $\frac{dh^{-1}(u)}{du} = \frac{d}{du}(\sqrt{u}) = \frac{d}{du}(u^{1/2}) = \frac{1}{2}u^{1/2-1} = \frac{1}{2}u^{-1/2}$.

(cont'd next slide...)

The Jacobian Transformation Method

Re-doing Example 2:

Let X have the PDF $f_X(x) = 2x$, $0 \le x \le 1$.
Let $U = X^2$. Find the PDF of U.

Solution:

- Transformation: $h(x) = x^2$
- Inverse: $h^{-1}(u) = \sqrt{u}$
- Jacobian: $\frac{dh^{-1}(u)}{du} = \frac{1}{2}u^{-1/2}$
- Deriving the PDF of U using the formula in the theorem:

$$\begin{aligned} f_U(u) &= f_X\{h^{-1}(u)\}\left|\frac{dh^{-1}(u)}{du}\right| \\ &= 2(\sqrt{u})\left|\frac{1}{2}u^{-1/2}\right| \\ &= 1. \quad \text{This is the } U(0,1) \text{ PDF.} \end{aligned}$$

We arrived at the same PDF as the CDF method's.

The Jacobian Transformation Method

Re-doing Example 3:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = \sigma Z + \mu$. Find the PDF of Y.

Solution:

- Domain of Z: $(-\infty, \infty)$ given
- Codomain of Y: $(-\infty, \infty)$
- Transformation: $y = h(z) = \sigma z + \mu$ given
- Deriving the inverse of the transformation, $h^{-1}(y)$:

  (write z in terms of y)

$$\begin{aligned} y &= \sigma z + \mu \\ \frac{y - \mu}{\sigma} &= z \end{aligned}$$

  Therefore, the inverse function is $h^{-1}(y) = \frac{y-\mu}{\sigma}$
- Jacobian: $\frac{dh^{-1}(y)}{dy} = \frac{d}{dy}\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma}$.

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The Jacobian Transformation Method

Re-doing Example 3:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = \sigma Z + \mu$. Find the PDF of Y.

Solution:

- Transformation: $h(z) = \sigma z + \mu$
- Inverse: $h^{-1}(y) = \frac{y-\mu}{\sigma}$
- Jacobian: $\frac{dh^{-1}(y)}{dy} = \frac{1}{\sigma}$
- Deriving the PDF of Y using the formula in the theorem:

$$
\begin{aligned}
f_Y(y) &= f_Z\{h^{-1}(y)\}\left|\frac{dh^{-1}(y)}{dy}\right| \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}} \left|\frac{1}{\sigma}\right| \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}.
\end{aligned}
$$

This is the Gaussian PDF. □

We arrived at the same PDF as the CDF method's.

Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Domain of Z: $(-\infty, \infty)$ given
- Codomain of Y: $(0, \infty)$
- Transformation: $y = h(z) = z^2$ is monotone on $(-\infty, 0)$ and $(0, \infty)$ given
- Deriving the inverse of the transformation, $h^{-1}(y)$: (write z in terms of y)

$$\begin{aligned} y &= z^2 \\ \pm\sqrt{y} &= z \end{aligned}$$

  Therefore, the inverse function is $h_1^{-1}(y) = \sqrt{y}$ if $z \geq 0$ and $h_2^{-1}(y) = -\sqrt{y}$ if $z < 0$.
- Jacobian: $\frac{dh_1^{-1}(y)}{dy} = \frac{d}{dy}\left(\sqrt{y}\right) = \frac{1}{2}y^{-1/2}$ if $z \geq 0$ and $\frac{dh_2^{-1}(y)}{dy} = \frac{d}{dy}\left(\sqrt{y}\right) = -\frac{1}{2}y^{-1/2}$ if $z < 0$.

(cont'd next slide...)

The Jacobian Transformation Method

Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.

Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Transformation: $h(z) = z^2$
- Inverse: $h_1^{-1}(y) = \sqrt{y}$ if $z \geq 0$ and $h_2^{-1}(y) = -\sqrt{y}$ if $z < 0$
- Jacobian: $\frac{dh_1^{-1}(y)}{dy} = \frac{1}{2} y^{-1/2}$ if $z \geq 0$ and $\frac{dh_2^{-1}(y)}{dy} = -\frac{1}{2} y^{-1/2}$ if $z < 0$
- Deriving the PDF of Y using the formula in the theorem (non-monotone):

$$\begin{aligned} f_Y(y) &= f_Z\{h_1^{-1}(y)\} \left| \frac{dh_1^{-1}(y)}{dy} \right| + f_Z\{h_2^{-1}(y)\} \left| \frac{dh_2^{-1}(y)}{dy} \right| \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \left| \frac{1}{2} y^{-1/2} \right| + \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{y})^2}{2}} \left| -\frac{1}{2} y^{-1/2} \right| \\ &= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}. \end{aligned}$$

formula for non-monotone functions

This is the $\chi^2(1)$ PDF. □

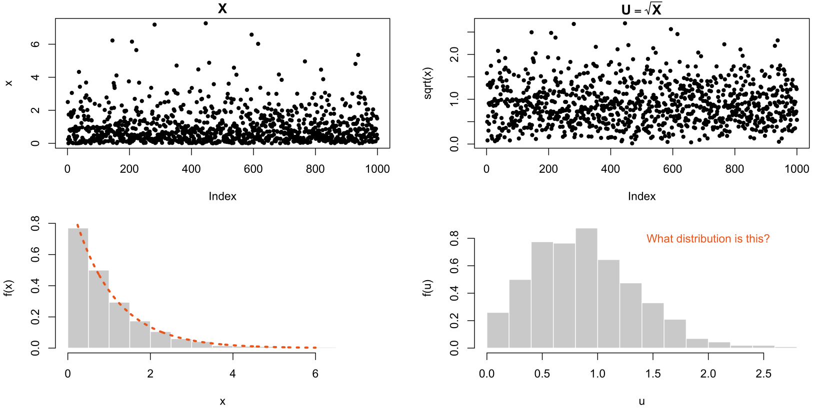
We arrived at the same PDF as the CDF method's.

The Jacobian Transformation Method

Example 5:

Suppose we have an exponential RV X with PDF $f_X(x) = e^{-x}$, $x \geq 0$. Let $U = \sqrt{X}$. Find the PDF of U.

Visualizing the problem...



The Jacobian Transformation Method

Example 5:

Suppose we have an exponential RV X with PDF $f_X(x) = e^{-x}$, $x \geq 0$. Let $U = \sqrt{X}$. Find the PDF of U.

Solution:

- Domain of X: $[0, \infty]$ given
- Codomain of U: $[0, \infty]$
- Transformation function: $u = h(x) = \sqrt{x}$ given
- Deriving the inverse of the transformation, $h^{-1}(u)$:

  (write x in terms of u)

$$\begin{aligned} u &= \sqrt{x} \\ u^2 &= x. \end{aligned}$$

  Therefore, the inverse function is $h^{-1}(u) = u^2$

- Jacobian: $\frac{dh^{-1}(u)}{du} = \frac{d}{du}(u^2) = 2u.$

(cont'd next slide...)

The Jacobian Transformation Method

Example 5:

Suppose we have an exponential RV X with PDF $f_X(x) = e^{-x}$, $x \geq 0$. Let $U = \sqrt{X}$. Find the PDF of U.

Solution:

- Transformation function: $h(x) = \sqrt{x}$
- Inverse: $h^{-1}(u) = u^2$
- Jacobian: $\frac{dh^{-1}(u)}{du} = 2u$
- Deriving the PDF of U using the formula in the theorem:

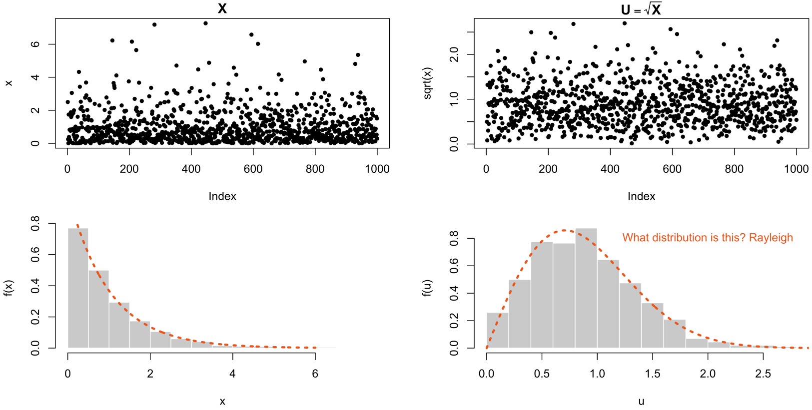
$$\begin{aligned} f_U(u) &= f_X\{h^{-1}(u)\}\left|\frac{dh^{-1}(u)}{du}\right| \\ &= e^{-u^2}\left|2u\right| \\ &= 2ue^{-u^2}. \quad \text{$2u$ is always positive since $u \in [0, \infty]$.} \end{aligned}$$

This is the Rayleigh distribution.

Example 5:

Suppose we have an exponential RV X with PDF $f_X(x) = e^{-x}$, $x \geq 0$.
Let $U = \sqrt{X}$. Find the PDF of U.

Visualizing the problem...



The MGF Method

The MGF Method

Theorem: Uniqueness Theorem

Let $m_X(t)$ and $m_Y(t)$ denote the MGF of RVs X and Y, respectively. If both MGFs exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

The MGF Method: How It Works

To find the distribution of the transformation:

1. Derive the MGF of the transformed RV.
2. Compare the MGF of the transformed RV to the MGFs of known distributions.
3. The distribution of the transformed RV follows the distribution of the matching MGF.

The MGF Method

Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.

Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Derive the MGF of the transformed RV.

$$
\begin{aligned}
m_Y(t) = m_{Z^2}(t) &= E(e^{tZ^2}) && \text{def'n of MGF} \\
&= \int_{-\infty}^{\infty} e^{tz^2} f(z) dz && \text{def'n of expected value} \\
&= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz && Z \text{ is standard normal} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2\left(\frac{1-2t}{2}\right)} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2\left(\frac{1}{1-2t}\right)}} dz
\end{aligned}
$$

the integrand resembles a Gaussian PDF with $\mu = 0$ and $\sigma^2 = \dfrac{1}{1-2t}$

(cont'd next slide...)

Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Derive the MGF of the transformed RV. (cont'd)

$$
\begin{aligned}
m_Y(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2\left(\frac{1}{1-2t}\right)}} dz \quad \text{Recall the Gaussian PDF: } \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \\
&= \sqrt{\frac{1}{1-2t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\frac{1}{1-2t}}} e^{-\frac{z^2}{2\left(\frac{1}{1-2t}\right)}} dz \quad \text{multiply a factor of 1} \\
&= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{1}{1-2t}}} e^{-\frac{z^2}{2\left(\frac{1}{1-2t}\right)}} dz \\
&= \frac{1}{\sqrt{1-2t}} (1) \quad \text{Gaussian PDF integrates to 1}
\end{aligned}
$$

(cont'd next slide...)

The MGF Method

Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.

Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Derive the MGF of the transformed RV. (cont'd)

$$m_Y(t) = \frac{1}{(1-2t)^{1/2}}$$

Recall Gamma MGF: $\frac{1}{(1-\beta t)^{\alpha}}$

Note that this is the MGF of a Gamma RV with $\alpha = 1/2$ and $\beta = 2$.

Thus, Y is χ^2 with $\nu = 1$ degree of freedom.

Questions?

Homework Exercises: 6.15, 6.20, 6.23, 6.28, 6.46

Solutions will be discussed this Friday by the TA.