STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 19: Functions of Random Variables (Multivariate)

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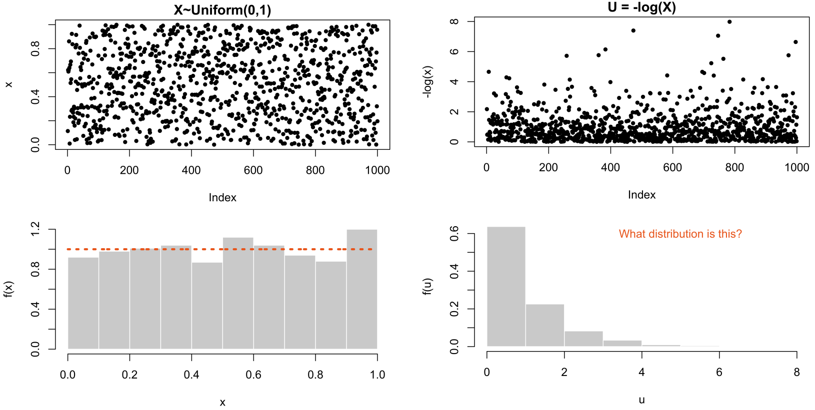
Outline

Previously...

Functions of Random Variables (Univariate)

Original RV

Transformed RV



Functions of Random Variables (Univariate)

- Suppose we have a random variable X with PDF $f_X(x)$.
- Define a new random variable $U = h(X)$, where h is a (one-to-one) monotone function.
  - $h(x) = e^x$
  - $h(x) = \ln(x)$
  - $h(x) = \sqrt{x}$
  - $h(x) = x^2$
- What is the PDF of U?
  - CDF Method
  - Jacobian Method (PDF-to-PDF Method or change of variable)
  - MGF Method

The CDF Method

Suppose $U = h(X)$, where the original RV X has PDF $f_X(x)$.

- If h is an increasing function, the CDF of U is

$$F_U(u) = F_X\{h^{-1}(u)\}.$$

- If h is a decreasing function, the CDF of U is

$$F_U(u) = 1 - F_X\{h^{-1}(u)\}.$$

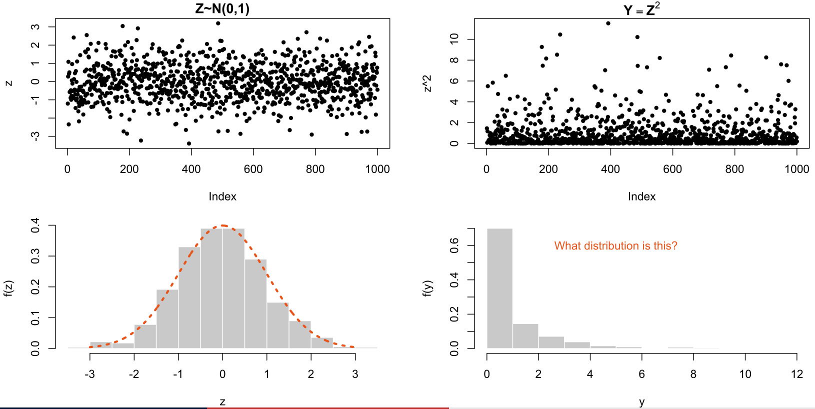
- The PDF of U, $f_U(u)$ can be obtained by differentiation as follows:

$$f_U(u) = \frac{d}{du} F_U(u).$$

Lec 18 Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Visualizing the problem...



The CDF Method

Lec 18 Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty \leq z \leq \infty$.

Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Domain of Z: $(-\infty, \infty)$
- Codomain of Y: $(0, \infty)$

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) && \text{CDF def'n} \\
&= P(Z^2 \leq y) && \text{transformation} \\
&= P\left(-\sqrt{y} \leq Z \leq \sqrt{y}\right) && \text{isolate the original RV} \\
&= P\left(Z \leq \sqrt{y}\right) - P\left(Z \leq -\sqrt{y}\right) \\
&= \Phi\left(\sqrt{y}\right) - \Phi\left(-\sqrt{y}\right). && \text{CDF of the original RV: Standard Normal CDF } \Phi(z)
\end{aligned}
$$

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The CDF Method

Lec 18 Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.

Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Domain of Z: $(-\infty, \infty)$
- Codomain of Y: $(0, \infty)$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}\{F_Y(y)\} = \frac{d}{dy}\{\Phi(\sqrt{y})\} - \frac{d}{dy}\{\Phi(-\sqrt{y})\} \\ &= \frac{1}{2} y^{-1/2}\{\phi(\sqrt{y})\} - \left[-\frac{1}{2} y^{-1/2}\{\phi(-\sqrt{y})\}\right] \end{aligned}$$

derivative of CDF $\Phi(z)$ is PDF $\phi(z)$; chain rule

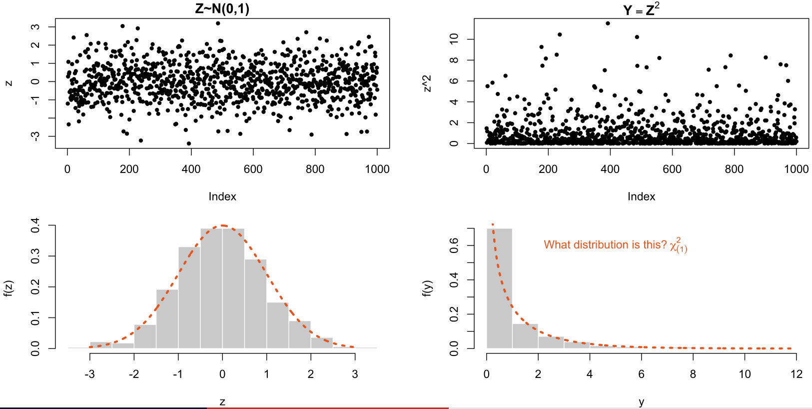
$$\begin{aligned} &= \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \\ &= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}. \quad \square \end{aligned}$$

This is the $\chi^2(1)$ PDF.

Lec 18 Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Visualizing the problem...



The Jacobian Transformation Method

Suppose $U = h(X)$, where the original RV X has PDF $f_X(x)$.

- The PDF of U, $f_U(u)$ can be obtained as follows:

$$f_U(u) = f_X\{h^{-1}(u)\}\left|\frac{dh^{-1}(u)}{du}\right|,$$

  where $|\cdot|$ is the absolute value function and $\frac{dh^{-1}(u)}{du}$ is called the Jacobian of the transformation.
- We need the following to obtain the new PDF:
  - original PDF: $f_X(x)$
  - transformation function: $u = h(x)$
  - inverse of the transformation: $x = h^{-1}(u)$
  - Jacobian: $\frac{dh^{-1}(u)}{du}$

Lec 18 Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.

Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Domain of Z: $(-\infty, \infty)$ given
- Codomain of Y: $(0, \infty)$
- Transformation: $y = h(z) = z^2$ is monotone on $(-\infty, 0)$ and $(0, \infty)$ given
- Deriving the inverse of the transformation, $h^{-1}(y)$: (write z in terms of y)

$$\begin{aligned} y &= z^2 \\ \pm\sqrt{y} &= z \end{aligned}$$

  Therefore, the inverse function is $h_1^{-1}(y) = \sqrt{y}$ if $z \geq 0$ and $h_2^{-1}(y) = -\sqrt{y}$ if $z < 0$.

- Jacobian: $\frac{dh_1^{-1}(y)}{dy} = \frac{d}{dy}\left(\sqrt{y}\right) = \frac{1}{2}y^{-1/2}$ if $z \geq 0$ and $\frac{dh_2^{-1}(y)}{dy} = \frac{d}{dy}\left(-\sqrt{y}\right) = -\frac{1}{2}y^{-1/2}$ if $z < 0$.

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The Jacobian Transformation Method

Lec 18 Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.

Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Transformation: $h(z) = z^2$
- Inverse: $h_1^{-1}(y) = \sqrt{y}$ if $z \geq 0$ and $h_2^{-1}(y) = -\sqrt{y}$ if $z < 0$
- Jacobian: $\frac{dh_1^{-1}(y)}{dy} = \frac{1}{2} y^{-1/2}$ if $z \geq 0$ and $\frac{dh_2^{-1}(y)}{dy} = -\frac{1}{2} y^{-1/2}$ if $z < 0$
- Deriving the PDF of Y using the formula in the theorem (non-monotone):

$$
\begin{aligned}
f_Y(y) &= f_Z\{h_1^{-1}(y)\} \left| \frac{dh_1^{-1}(y)}{dy} \right| + f_Z\{h_2^{-1}(y)\} \left| \frac{dh_2^{-1}(y)}{dy} \right| \\
&= \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \left| \frac{1}{2} y^{-1/2} \right| + \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{y})^2}{2}} \left| -\frac{1}{2} y^{-1/2} \right|}_{\text{formula for non-monotone functions}} \\
&= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}. \quad \text{This is the } \chi^2(1) \text{ PDF.} \quad \square
\end{aligned}
$$

We arrived at the same PDF as the CDF method's.

The MGF Method

Suppose $U = h(X)$, where the original RV X has PDF $f_X(x)$.

To find the distribution of the transformation:

1. Derive the MGF of the transformed RV.
2. Compare the MGF of the transformed RV to the MGFs of known distributions.
3. The distribution of the transformed RV follows the distribution of the matching MGF.

The MGF Method

Lec 18 Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Derive the MGF of the transformed RV.

$$
\begin{aligned}
m_Y(t) = m_{Z^2}(t) &= E(e^{tZ^2}) \quad \text{def'n of MGF} \\
&= \int_{-\infty}^{\infty} e^{tz^2} f(z) dz \quad \text{def'n of expected value} \\
&= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \quad Z \text{ is standard normal} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2\left(\frac{1-2t}{2}\right)} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2\left(\frac{1}{1-2t}\right)}} dz
\end{aligned}
$$

the integrand resembles a Gaussian PDF with $\mu = 0$ and $\sigma^2 = \frac{1}{1-2t}$

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Lec 18 Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Derive the MGF of the transformed RV. (cont'd)

$$m_Y(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2\left(\frac{1}{1-2t}\right)}} dz$$

Recall the Gaussian PDF: $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$

$$= \sqrt{\frac{1}{1-2t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\frac{1}{1-2t}}} e^{-\frac{z^2}{2\left(\frac{1}{1-2t}\right)}} dz$$

multiply a factor of 1

$$= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{1}{1-2t}}} e^{-\frac{z^2}{2\left(\frac{1}{1-2t}\right)}} dz$$

$$= \frac{1}{\sqrt{1-2t}} (1)$$

Gaussian PDF integrates to 1

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Lec 18 Re-doing Example 4:

Let Z have the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty \leq z \leq \infty$.
Let $Y = Z^2$. Find the PDF of Y.

Solution:

- Derive the MGF of the transformed RV. (cont'd)

$$m_Y(t) = \frac{1}{(1-2t)^{1/2}}$$

Recall Gamma MGF: $\frac{1}{(1-\beta t)^{\alpha}}$

Note that this is the MGF of a Gamma RV with $\alpha = 1/2$ and $\beta = 2$.

Thus, Y is χ^2 with $\nu = 1$ degree of freedom.

Functions of Random Variables (Multivariate)

Functions of Random Variables (Multivariate)

- Suppose we have two random variables X_1 and X_2 with joint PDF $f_{X_1,X_2}(x_1, x_2)$.
- Define new random variables $U_1 = h_1(X_1, X_2)$ and $U_2 = h_2(X_1, X_2)$, where h_1 and h_2 are (one-to-one) monotone functions.
- What is the joint PDF of U_1 and U_2?
  - Jacobian Method (PDF-to-PDF Method or change of variable)
  - MGF Method

The Jacobian Transformation Method

The Jacobian Transformation Method

Theorem: The PDF-to-PDF Method (monotone)

Suppose that X_1 and X_2 are continuous RVs with joint PDF $f_{X_1,X_2}(x_1, x_2)$ and that for all (x_1, x_2) such that $f_{X_1,X_2}(x_1, x_2) > 0$,

$$u_1 = h_1(x_1, x_2) \quad \text{and} \quad u_2 = h_2(x_1, x_2),$$

are one-to-one transformations from (x_1, x_2) to (u_1, u_2) with inverse

$$x_1 = h_1^{-1}(u_1, u_2) \quad \text{and} \quad x_2 = h_2^{-1}(u_1, u_2).$$

If $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1 and u_2, and the determinant of the Jacobian matrix is not equal to 0, then the joint PDF of $U_1 = h_1(X_1, X_2)$ and $U_2 = h_2(X_1, X_2)$ is

$$f_{U_1,U_2}(u_1, u_2) = f_{X_1,X_2}\{h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)\}|J|,$$

where $|J|$ is the absolute value of the Jacobian.

The Jacobian Transformation Method: How it Works

- We need the following to obtain the new joint PDF:
  - original joint PDF: $f_{X_1,X_2}(x_1, x_2)$
  - transformations: $u_1 = h_1(x_1, x_2)$ and $u_2 = h_2(x_1, x_2)$
  - inverse of the transformation: $x_1 = h_1^{-1}(u_1, u_2)$ and $x_2 = h_2^{-1}(u_1, u_2)$
  - Jacobian: (determinant of the matrix of partial derivatives)

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial h_1^{-1}(u_1,u_2)}{\partial u_2} \\ \frac{\partial h_2^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial h_2^{-1}(u_1,u_2)}{\partial u_2} \end{vmatrix},$$

  where $|\cdot|$ takes the determinant of the Jacobian matrix.

The Jacobian Transformation Method

Example 1:

Let X_1 and X_2 have the joint PDF

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} e^{-(x_1+x_2)}, & \text{for } x_1 \geq 0, x_2 \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Consider two RVs U_1 and U_2 defined in the following manner:

$$U_1 = X_1 + X_2 \quad \text{and} \quad U_2 = \frac{X_1}{X_1 + X_2}.$$

Find the joint PDF of U_1 and U_2.

The Jacobian Transformation Method

Solution:

- Step 1: Identify the transformation. (new RVs = function of original RVs)

$$\begin{cases} U_1 = X_1 + X_2 \\ U_2 = \frac{X_1}{X_1+X_2} \end{cases} \Rightarrow \begin{cases} u_1 = h_1(x_1, x_2) = x_1 + x_2 \\ u_2 = h_2(x_1, x_2) = \frac{x_1}{x_1+x_2} \end{cases}$$

  This means that if $x_1 \geq 0, x_2 \geq 0$, then $u_1 \geq 0$ and $0 \leq u_2 \leq 1$.

- Step 2: Deriving the inverse transformations, $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$. (original RVs = function of new RVs)

$$\begin{cases} U_1 = X_1 + X_2 \\ U_2 = \frac{X_1}{X_1+X_2} \end{cases} \Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ U_2 = \frac{X_1}{X_1+X_2} \end{cases} \Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ U_2 = \frac{U_1 - X_2}{U_1 - X_2 + X_2} \end{cases}$$

$$\Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ U_2 = \frac{U_1 - X_2}{U_1} \end{cases} \Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ U_1 U_2 = U_1 - X_2 \end{cases} \Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ X_2 = U_1 - U_1 U_2 \end{cases}$$

$$\Rightarrow \begin{cases} X_1 = U_1 - (U_1 - U_1 U_2) \\ X_2 = U_1 - U_1 U_2 \end{cases} \Rightarrow \begin{cases} X_1 = U_1 U_2 \\ X_2 = U_1 - U_1 U_2. \end{cases}$$

  Thus, the inverse transformations are $\begin{cases} x_1 = h_1^{-1}(u_1, u_2) = u_1 u_2 \\ x_2 = h_2^{-1}(u_1, u_2) = u_1 - u_1 u_2. \end{cases}$

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The Jacobian Transformation Method

Solution:

- Step 3: Obtain the Jacobian of the inverse transformations:
  $$\begin{cases} h_1^{-1}(u_1, u_2) = u_1 u_2 \\ h_2^{-1}(u_1, u_2) = u_1 - u_1 u_2. \end{cases}$$

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial h_1^{-1}(u_1,u_2)}{\partial u_2} \\ \frac{\partial h_2^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial h_2^{-1}(u_1,u_2)}{\partial u_2} \end{vmatrix} = \begin{vmatrix} u_2 & u_1 \\ 1 - u_2 & -u_1 \end{vmatrix}$$
$$= \{-u_2 u_1 - u_1(1 - u_2)\} = -u_1.$$

Recall determinant of a matrix formula: $ad - bc$

- Step 4: Apply the formula
  - (Given) original joint PDF: $f_{X_1,X_2}(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & \text{for } x_1 \geq 0, x_2 \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$

$$\begin{aligned} f_{U_1,U_2}(u_1, u_2) &= f_{X_1,X_2}\{h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)\}|J| \\ &= e^{-(u_1 u_2 + u_1 - u_1 u_2)}|-u_1| \\ &= u_1 e^{-u_1}, \quad u_1 \geq 0, \quad 0 \leq u_2 \leq 1. \quad \square \end{aligned}$$

The Jacobian Transformation Method

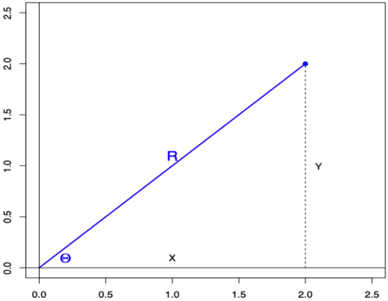
Example 2: (Cartesian to Polar Transformation)
Let X and Y be independent standard normal random variables.

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}, \quad -\infty < x, y < \infty.$$

Consider the polar transformation:

$X = R\cos\Theta \quad \text{and} \quad Y = R\sin\Theta.$

Find the joint PDF of R and Θ.



The Jacobian Transformation Method

Solution:

- Step 1: Identify the transformation. (new RVs = function of original RVs)
  - Solving for R:

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases} \Rightarrow \begin{cases} X^2 = R^2\cos^2\Theta \\ Y^2 = R^2\sin^2\Theta \end{cases} \Rightarrow X^2 + Y^2 = R^2(\cos^2\Theta + \sin^2\Theta)$$

$$\Rightarrow X^2 + Y^2 = R^2 \Rightarrow R = \sqrt{X^2 + Y^2}.$$

  - Solving for Θ:

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases} \Rightarrow \frac{Y}{X} = \frac{\sin\Theta}{\cos\Theta} \Rightarrow \frac{Y}{X} = \tan\Theta \Rightarrow \Theta = \tan^{-1}\left(\frac{Y}{X}\right).$$

Therefore, the transformation is $\begin{cases} r = h_1(x_1, x_2) = \sqrt{x^2 + y^2} \\ \theta = h_2(x_1, x_2) = \tan^{-1}\left(\frac{y}{x}\right). \end{cases}$

This means that if $-\infty < x, y < \infty$, then $0 < \theta < 2\pi$ and $0 < r < \infty$.

- Step 2: Deriving the inverse transformations, $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$. (original RVs = function of new RVs)

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases} \text{(given)} \Rightarrow \begin{cases} x = h_1^{-1}(r, \theta) = r\cos\theta \\ y = h_2^{-1}(r, \theta) = r\sin\theta. \end{cases}$$

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The Jacobian Transformation Method

Solution:

- Step 3: Obtain the Jacobian of the inverse transformations:
  $$\begin{cases} h_1^{-1}(r,\theta) = r\cos\theta \\ h_2^{-1}(r,\theta) = r\sin\theta. \end{cases}$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial h_1^{-1}(r,\theta)}{\partial r} & \frac{\partial h_1^{-1}(r,\theta)}{\partial \theta} \\ \frac{\partial h_2^{-1}(r,\theta)}{\partial r} & \frac{\partial h_2^{-1}(r,\theta)}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = r. \end{aligned}$$

Recall determinant of a matrix formula: $ad - bc$

- Step 4: Apply the formula
  - (Given) original joint PDF: $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}, \quad -\infty < x, y < \infty.$

$$\begin{aligned} f_{R,\Theta}(r,\theta) &= f_{X,Y}\{h_1^{-1}(r,\theta), h_2^{-1}(r,\theta)\}|J| \\ &= \frac{1}{2\pi} e^{-\{(r\cos\theta)^2 + (r\sin\theta)^2\}/2}|r| \\ &= \frac{1}{2\pi} r e^{-r^2/2}, \quad 0 < \theta < 2\pi, \quad 0 < r < \infty. \quad \square \end{aligned}$$

The Jacobian Transformation Method

Example 3: (Sum of Two Uniform RVs)
Let X_1 and X_2 be independent and identical $\mathcal{U}(0,1)$ random variables.
Find the PDF of $Y_1 = X_1 + X_2$.
Solution:

- The Jacobian method requires us to define a second transformation variable.
- Let $Y_2 = X_2$.
- Step 1: Identify the transformation. (new RVs = function of original RVs)

  $$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{cases} \Rightarrow \begin{cases} y_1 = h_1(x_1, x_2) = x_1 + x_2 \\ y_2 = h_2(x_1, x_2) = x_2 \end{cases}$$

  This means that if $0 \le x_1 \le 1, 0 \le x_2 \le 1$, then

  - $0 \le x_1 + x_2 \le 2 \Rightarrow \underline{0 \le y_1 \le 2}$.
  - $0 \le x_2 \le 1 \Rightarrow \underline{0 \le y_2 \le 1}$.
  - Since $y_1 = x_1 + x_2 \Rightarrow x_1 = y_1 - x_2 \Rightarrow x_1 = y_1 - y_2$, then
    $0 \le y_1 - y_2 \le 1 \Rightarrow \underline{y_2 \le y_1 \le 1 + y_2}$.

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The Jacobian Transformation Method

Solution:

- Step 2: Deriving the inverse transformations, $h_1^{-1}(y_1, y_2)$ and $h_2^{-1}(y_1, y_2)$.
  (original RVs = function of new RVs)

$$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{cases} \Rightarrow \begin{cases} Y_1 = X_1 + Y_2 \\ X_2 = Y_2 \end{cases} \Rightarrow \begin{cases} X_1 = Y_1 - Y_2 \\ X_2 = Y_2. \end{cases}$$

  Thus, the inverse transformations are $\begin{cases} x_1 = h_1^{-1}(y_1, y_2) = y_1 - y_2 \\ x_2 = h_2^{-1}(y_1, y_2) = y_2. \end{cases}$

- Step 3: Obtain the Jacobian of the inverse transformations:

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial h_1^{-1}(y_1,y_2)}{\partial y_2} \\ \frac{\partial h_2^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial h_2^{-1}(y_1,y_2)}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = (1)(1) - (-1)(0) = 1.$$

- Step 4: Apply the formula
  - Since X_1 and X_2 be independent and identical $\mathcal{U}(0, 1)$,

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = (1)(1) = 1.$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\{h_1^{-1}(y_1, y_2), h_2^{-1}(y_1, y_2)\}|J| = (1)|1| = 1,$$

$y_2 \le y_1 \le 1 + y_2,\ 0 \le y_1 \le 2,\ 0 \le y_2 \le 1.$

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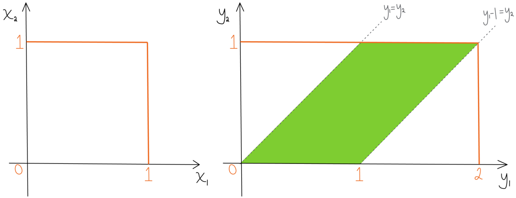
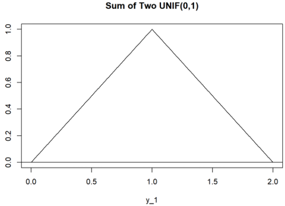
The Jacobian Transformation Method

Solution:

- Joint PDF: $f_{Y_1,Y_2}(y_1,y_2) = 1, \quad y_2 \le y_1 \le 1 + y_2,\ 0 \le y_1 \le 2,\ 0 \le y_2 \le 1.$



- Step 5: Obtain the marginal of Y_1.
  - If $0 < y_1 < 1$, then
    $f_{Y_1}(y_1) = \int_0^{y_1} f_{Y_1,Y_2}(y_1,y_2)dy_2 = \int_0^{y_1} 1 dy_2 = y_2\big|_0^{y_1} = y_1.$
  - If $1 \le y_1 < 2$, then
    $f_{Y_1}(y_1) = \int_{y_1-1}^{1} f_{Y_1,Y_2}(y_1,y_2)dy_2 = \int_{y_1-1}^{1} 1 dy_2 = y_2\big|_{y_1-1}^{1} = 1-(y_1-1) = 2-y_1.$ □



The MGF Method

The MGF Method

Theorem: MGF of a Sum of Independent RVs

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with MGFs $m_{Y_1}(t), m_{Y_2}(t), \ldots, m_{Y_n}(t)$, respectively. If $U = Y_1 + Y_2 + \ldots + Y_n$, then

$$m_U(t) = m_{Y_1}(t) m_{Y_2}(t) \cdots m_{Y_n}(t)$$

Proof:

$$\begin{aligned}
m_U(t) &= E\left(e^{tU}\right) && \text{def'n of MGF} \\
&= E\left\{e^{t(Y_1+Y_2+\ldots+Y_n)}\right\} && \text{given: } U = Y_1 + Y_2 + \ldots + Y_n \\
&= E\left(e^{tY_1} e^{tY_2} \cdots + e^{tY_n}\right) \\
&= E\left(e^{tY_1}\right) E\left(e^{tY_2}\right) \cdots E\left(e^{tY_n}\right) && \text{independence} \\
&= m_{Y_1}(t) m_{Y_2}(t) \cdots m_{Y_n}(t). && \text{def'n of MGF} \quad \square
\end{aligned}$$

Theorem: Sum of Independent Gaussian RVs

Let $Y_1, Y_2, \ldots, Y_n$ be independent Gaussian RVs with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for $i = 1, 2, \ldots, n$, and let $a_1, a_2, \ldots, a_n$ be constants. If

$$U = \sum_{i=1}^{n} a_i Y_i = a_1 Y_1 + a_2 Y_2 + \ldots + a_n Y_n,$$

then U is a Gaussian RV with

$$E(U) = \sum_{i=1}^{n} a_i \mu_i = a_1 \mu_1 + a_2 \mu_2 + \ldots + a_n \mu_n,$$

and

$$V(U) = \sum_{i=1}^{n} a_i^2 \sigma_i^2 = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \ldots + a_n^2 \sigma_n^2.$$

The MGF Method

Proof:

$$m_{a_i Y_i}(t) = E\left(e^{ta_i Y_i}\right) \quad \text{def'n of MGF}$$
$$= m_{Y_i}(a_i t) \quad \text{MGF of linear transformation}$$
$$= e^{\mu_i a_i t + \frac{\sigma_i^2 a_i^2 t^2}{2}}. \quad Y_i \text{ is Gaussian with mean } \mu_i \text{ and variance } \sigma_i^2\text{; MGF Gaussian RV}$$

$$m_U(t) = m_{\sum_{i=1}^n a_i Y_i}(t)$$
$$= m_{a_1 Y_1}(t) m_{a_2 Y_2}(t) \cdots m_{a_n Y_n}(t) \quad \text{Theorem on MGF of a Sum of Independent RVs}$$
$$= e^{\mu_1 a_1 t + \frac{\sigma_1^2 a_1^2 t^2}{2}} e^{\mu_2 a_2 t + \frac{\sigma_2^2 a_2^2 t^2}{2}} \cdots e^{\mu_n a_n t + \frac{\sigma_n^2 a_n^2 t^2}{2}} \quad \text{MGF of each } a_i Y_i$$
$$= e^{t \sum_{i=1}^n a_i \mu_i + \frac{t^2}{2} \sum_{i=1}^n a_i^2 \sigma_i^2}.$$

Matching the MGF above to the list of popular MGFs, this is the MGF of a Gaussian RV with mean $\sum_{i=1}^n a_i \mu_i$ and variance $\sum_{i=1}^n a_i^2 \sigma_i^2$.

Therefore, $U \sim \mathcal{N}\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$. □

The MGF Method

Example 4:

Define $V = \sum_{i=1}^{n} Z_i^2$, where Z_i, $i = 1, \ldots, n$ are independent and identical $\mathcal{N}(0, 1)$ RVs. Find the PDF of V.

$$
\begin{aligned}
m_V(t) &= m_{\sum_{i=1}^{n} Z_i^2}(t) \\
&= m_{Z_1^2}(t) m_{Z_2^2}(t) \cdots m_{Z_n^2}(t) \quad \text{Theorem on MGF of a Sum of Independent RVs} \\
&= \frac{1}{(1-2t)^{1/2}} \frac{1}{(1-2t)^{1/2}} \cdots \frac{1}{(1-2t)^{1/2}}
\end{aligned}
$$

Lec 18, Ex. 4: Z_i^2 is $\chi^2(1)$; MGF of $\chi^2(1) = \frac{1}{(1-2t)^{1/2}}$

$$
= \frac{1}{(1-2t)^{n/2}}.
$$

Matching the MGF above to the list of popular MGFs, this is the MGF of a Gamma RV with $\alpha = n/2$ and $\beta = 2$.

Therefore, $V \sim \chi^2_{(n)}$. □

Probability Integral Transform

Probability Integral Transform

Theorem: Probability Integral Transform

1. Let X have a continuous and strictly increasing CDF $F_X(x)$. Define $U = F_X(X)$. Then, $U \sim \mathcal{U}(0, 1)$.
2. Let $U \sim \mathcal{U}(0, 1)$ and let F be a continuous CDF with quantile function F^{-1}. Let $X = F^{-1}(U)$ Then, X has CDF $F(x)$.

- This theorem tells us how to generate random numbers from any distribution.
- If F^{-1} is available in closed form, we can simply generate uniform random numbers and then transform them using F^{-1}.
- Thus, much of the effort in generating random numbers is focused on generating uniform random numbers.

Probability Integral Transform

Theorem: Probability Integral Transform

1. Let X have a continuous and strictly increasing CDF $F_X(x)$. Define $U = F_X(X)$. Then, $U \sim \mathcal{U}(0,1)$.
2. Let $U \sim \mathcal{U}(0,1)$ and let F be a continuous CDF with quantile function F^{-1}. Let $X = F^{-1}(U)$ Then, X has CDF $F(x)$.

Proof of (1): (need to show that U is a $\mathcal{U}(0,1)$ RV)

$$\begin{aligned}
F_U(u) &= P\{F_X(X) \leq u\} && \text{def'n of CDF} \\
&= P\left[F_X^{-1}\{F_X(X)\} \leq F_X^{-1}(u)\right] && \text{apply inverse function to both sides} \\
&= P\left\{X \leq F_X^{-1}(u)\right\} && \text{def'n of inverse function: } f^{-1}\{f(x)\} = x \\
&= F_X\{F_X^{-1}(u)\} && \text{def'n of CDF} \\
&= u. && \text{def'n of inverse function: } f^{-1}\{f(x)\} = x
\end{aligned}$$

This is the CDF of a uniform RV over the interval $(0,1)$. Hence, $U \sim \mathcal{U}(0,1)$. □

This tells us that if we apply the CDF of a RV to itself, we will get a $\mathcal{U}(0,1)$ RV...

Probability Integral Transform

Theorem: Probability Integral Transform

1. Let X have a continuous and strictly increasing CDF $F_X(x)$. Define $U = F_X(X)$. Then, $U \sim \mathcal{U}(0,1)$.
2. Let $U \sim \mathcal{U}(0,1)$ and let F be a continuous CDF with quantile function F^{-1}. Let $X = F^{-1}(U)$ Then, X has CDF $F(x)$.

Proof of (2): (need to show that $P(X \leq x) = F(x)$)

$$
\begin{aligned}
P(X \leq x) &= P\left\{F^{-1}(U) \leq x\right\} && \text{given transformation} \\
&= P\left[F\left\{F^{-1}(U)\right\} \leq F(x)\right] && \text{apply function to both sides} \\
&= P\left\{U \leq F(x)\right\} && \text{def'n of inverse function: } f^{-1}\{f(x)\} = x \\
&= F_U\{F(x)\} && \text{def'n of CDF} \\
&= F(x). && \text{CDF of a uniform RV}
\end{aligned}
$$

□

This tells us that we can obtain any desired RV by applying the inverse CDF to a $\mathcal{U}(0,1)$ RV...

Probability Integral Transform

Example 5:

Find a transformation $G(U)$ such that if U has a uniform distribution on $(0, 1)$, the $G(U)$ has a uniform distribution on $(3, 5)$.

Solution:

- Define the new RV: Let $X = G(U)$.
- By the probability integral transform theorem, the CDF of X is $G^{-1}(x)$.
- We want X to be uniform on $(3, 5)$. given
- This means that the CDF of X has the form $G^{-1}(x) = \frac{x-3}{2}$.

  Lec 14, Slide 4: CDF of $\mathcal{U}(a, b)$: $F(x) = \begin{cases} 0, & x < \theta_1 \\ \frac{x-\theta_1}{\theta_2-\theta_1}, & \theta_1 \leq x \leq \theta_2 \\ 1, & x > \theta_2. \end{cases}$

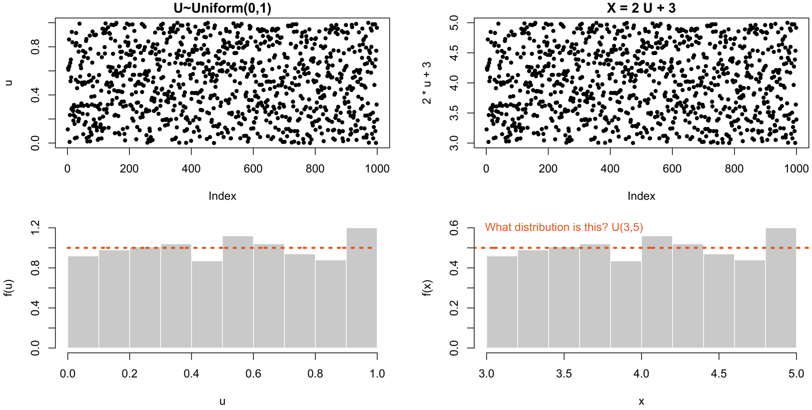
- To solve for the transformation $G(U)$, we need to find the inverse of $G^{-1}(x)$.
  - Let $u = \frac{x-3}{2}$.
  - Isolate x: $u = \frac{x-3}{2} \longrightarrow 2u = x - 3 \longrightarrow x = 2u + 3$.
  - Therefore, the required transformation of U is $G(U) = 2U + 3$. $\square$

Probability Integral Transform

Example 5:

Find a transformation $G(U)$ such that if U has a uniform distribution on $(0,1)$, the $G(U)$ has a uniform distribution on $(3,5)$.

Visualizing the problem...



Probability Integral Transform

Example 6:

Find a transformation $G(U)$ such that if U has a uniform distribution on $(0, 1)$, the $G(U)$ has an exponential distribution with $\beta = 1$.

Solution:

- Define the new RV: Let $X = G(U)$.
- By the probability integral transform theorem, the CDF of X is $G^{-1}(x)$.
- We want X to be Exp(1). given
- This means that the CDF of X has the form $G^{-1}(x) = 1 - e^{-x}$.

  Lec 14, Slide 5: CDF of exponential RV: $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & 0 \leq x < \infty \end{cases}$

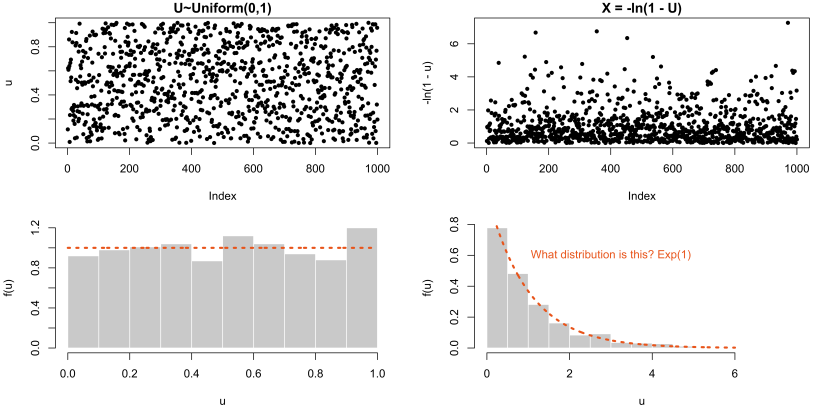
- To solve for the transformation $G(U)$, we need to find the inverse of $G^{-1}(x)$.
  - Let $u = 1 - e^{-x}$.
  - Isolate x: $u = 1 - e^{-x} \longrightarrow e^{-x} = 1 - u \longrightarrow x = -\ln(1 - u)$.
  - Therefore, the required transformation of U is $G(U) = -\ln(1 - U)$.

□

Example 6:

Find a transformation $G(U)$ such that if U has a uniform distribution on $(0,1)$, the $G(U)$ has an exponential distribution with $\beta = 1$.

Visualizing the problem...



Questions?

Homework Exercises: 6.15, 6.20, 6.23, 6.28, 6.46

Solutions will be discussed this Friday by the TA.