STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 2: Counting rules; Conditional probability; Independence

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Outline

Previously...

Review: Set Theory

- Definition: set - a collection of objects; denoted in capital letters

Type	Notation (Symbol)	Definition
universal set	S	the set containing all elements and of which all other sets are subsets
subset	$\subset$	a set whose elements are contained in another set
null set	$\emptyset$	a set that does not contain any elements
disjoint (or mutually exclusive) sets	$A \cap B = \emptyset$	two sets with no members or elements in common

- Operations

Given	Operation	Notation (Symbol)	Result
sets A, B	intersection	$A \cap B$	the set of all elements found in both A and B
sets A, B	union	$A \cup B$	the set of all elements found in one or both
subset C of S	complement	$\bar{C}$ or C^c or C'	the set of all elements in S but not in C

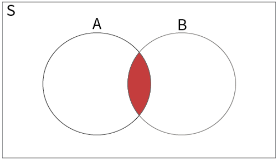


Figure: $A \cap B$

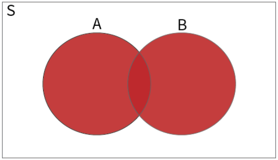


Figure: $A \cup B$

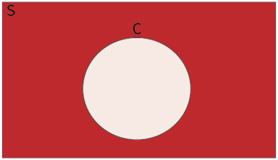
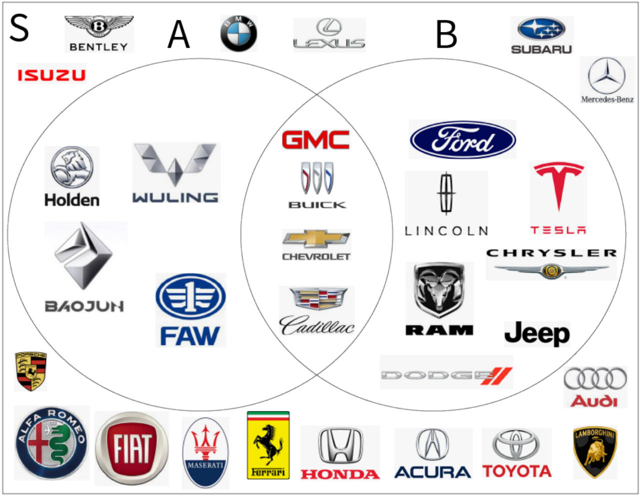


Figure: $\bar{C}$

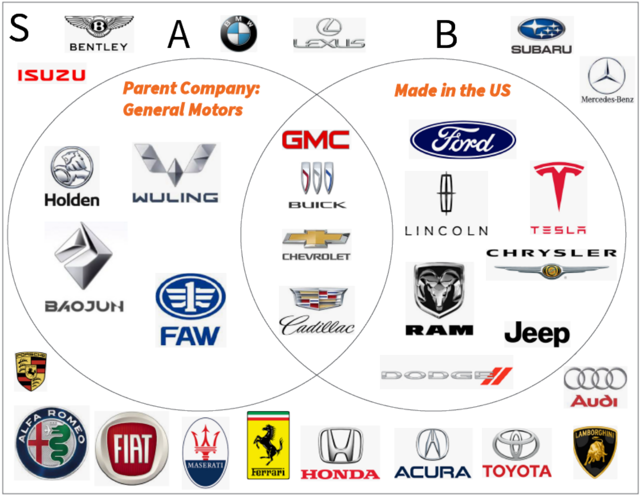
Review: Set Theory

What is S? A? B?



Review: Set Theory

What is S? A? B?



Source: www.alansfactoryoutlet.com

Review: Set Theory

We will need the concepts in set theory when computing probabilities...

Review: Probability

- commonly used to indicate one's belief in the occurrence of a future event
- Two ingredients in order to compute probability:
  1. sample space - the set of ALL possible outcomes for an activity or experiment
  2. outcome (also known as event) - a subset of the sample space whose probability we would like to compute

$$\text{probability} = \frac{\text{number of desired outcomes}}{\text{total number of all possible outcomes}} = \frac{\text{number of elements inside desired outcomes}}{\text{number of elements inside sample space}}$$

More Probability Theory

Experiments

- How do we know the sample space and the outcome?
- We need to conduct an *experiment*.
  What is an experiment?
  - It is a process by which an observation is made.
  - Any action for which all possible outcomes can be listed, but for which the actual outcome cannot be predicted.
  - Example: coin and die tossing, observing the stock price movements, observing the weather
- A result of an experiment is called an outcome.
- The set of all possible outcomes is called the sample space.
- An event is any combination of outcomes. It is a subset of the sample space to which we assign a probability.

Source: Shafer, D. S., & Zhang, Z. (2012). Introductory Statistics.

Source: Pishro-Nik, H., (2014). Introduction to probability, statistics, and random processes.

Experiments $\rightarrow$ Probability

Experiment	**Outcome**	**Sample Space** (S)	**Event** (A)	$P(A)$
toss a coin 3x	(H,H,H) or (H,H,T) or (H,T,H) or (T,H,H) or (H,T,T) or (T,H,T) or (T,T,H) or (T,T,T)	$S = \{(H,H,H), (H,H,T),(H,T,H), (T,H,H),(H,T,T), (T,H,T),(T,T,H), (T,T,T)\}$	same faces are showing up: $A = \{(H,H,H),(T,T,T)\}$	$\frac{2}{8} = 0.25$
roll a die	1 or 2 or 3 or 4 or 5 or 6	$S = \{1,2,3,4,5,6\}$	an even number is rolled: $A = \{2,4,6\}$	$\frac{3}{6} = 0.5$
draw a card from a deck of 52 cards	Ace of hearts or two of hearts or three of hearts, etc...	$S = \{$Ace of hearts, two of hearts, three of hearts,...$\}$ *(all the 52 cards)*	drawing a red face card: $A = \{$King of hearts, Queen of hearts, Jack of hearts, King of diamonds, Queen of diamonds, Jack of diamonds$\}$	$\frac{6}{52} \approx 0.12$

Ultimately, the goal is to compute the probabilities of events.

Events

Given	Operation	Notation (Symbol)	Result
events A, B	intersection	$A \cap B$	the event that both event A and event B happen
events A, B	union	$A \cup B$	the event that either event A happens or event B happens or both events happen
event C (in S)	complement	$\bar{C}$ or C^c or C'	the event that C does not happen; all outcomes in S that are not in C

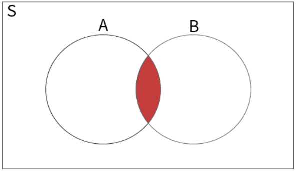


Figure: $A \cap B$

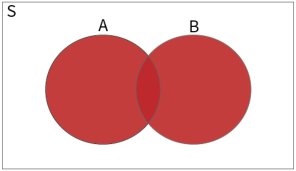


Figure: $A \cup B$

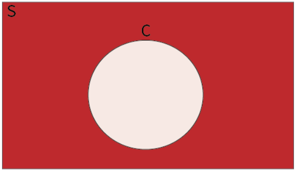


Figure: $\bar{C}$

Events

- Useful relationships of events
  - $A \cap \emptyset = \emptyset$
  - $A \cup \emptyset = A$
  - $A \cap \bar{A} = \emptyset$
  - $A \cup \bar{A} = S$
  - $\overline{(\overline{A})} = A$
  - $\overline{S} = \emptyset$
  - $\overline{\emptyset} = S$
  - $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (DeMorgan's law)
  - $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (DeMorgan's law)

Axiomatic Definition of Probability

Definition

Suppose S is a sample space associated with an experiment. To every event A in S (A is a subset of S), we assign a number, $P(A)$, called the *probability of* A, so that the following axioms hold:

Axiom 1: $P(A) \geq 0$. (*non-negativity*)
Axiom 2: $P(S) = 1$. (*unitary*)
Axiom 3: If $A_1, A_2, A_3, \dots$ form a sequence of pairwise mutually exclusive events in S (that is, $A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = \sum_{i=1}^{\infty} P(A_i).$$

Properties & Laws of Probability

- Some properties
  - Empty set: $P(\emptyset) = 0$
  - Monotonicity: If $A \subseteq B$, then $P(A) \leq P(B)$.
  - Numeric bound: $0 \leq P(A) \leq 1$

Theorem 2.6. The Additive Law of Probability

The probability of the union of two events A and B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive events, then $P(A \cap B) = 0$ and

$$P(A \cup B) = P(A) + P(B).$$

- Further properties
  - Complementary rule: $P(A) = 1 - P(\bar{A})$
  - Additive rule for 3 events: $P(A \cup B \cup C) = P(A) + P(B) + P(C)$
    $-P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

The Additive Law of Probability

Does this formula makes sense?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

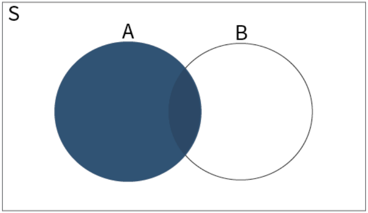


Figure: $P(A)$

The Additive Law of Probability

Does this formula makes sense?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

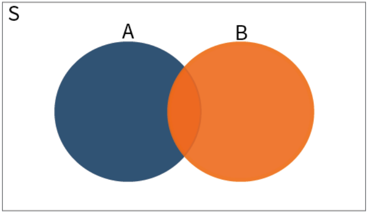


Figure: $P(A) + P(B)$

The probability of the intersection of A and B was counted twice. Remove one $P(A \cap B)$ to avoid double counting.

Counting Rules

Counting

Remember the formula for computing probabilities (with equally likely outcomes):

$$\text{probability} \quad = \quad \frac{\text{number of desired outcomes}}{\text{total number of all possible outcomes}}$$

In order to properly compute probabilities, one needs to learn to count...

Fundamental Counting Principle

If one thing can be done in m ways and another thing can be done in n ways, the two things can be done in mn ways...

Suppose a customer wants to purchase a tablet computer. The store is offering her a large or a small screen; a 64GB, 128GB, or 256GB storage capacity, and a black or white cover. How many different options does the customer have?

1. L-64-B
2. L-64-W
3. L-128-B
4. L-128-W
5. L-256-B
6. L-256-W
7. S-64-B
8. S-64-W
9. S-128-B
10. S-128-W
11. S-256-B
12. S-256-W

Thus, there are 12 available options for the customer.

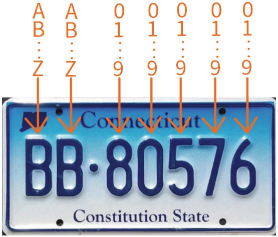
The same result can be obtained by simply multiplying the number of options in each category (screen size, memory, color) to get the total number of possibilities: $2 \times 3 \times 2 = 12$.

Fundamental Counting Principle

Theorem 2.1. Multiplication Principle

With m elements $a_1, a_2, \ldots, a_m$ and n elements $b_1, b_2, \ldots, b_n$, it is possible to form $mn = m \times n$ pairs containing one element from each group.

- Example 1: How many CT license plates, consisting of 2 letters followed by 5 digits are possible?



There are 26 letters and 10 digits. Therefore,

$$26 \times 26 \times 10 \times 10 \times 10 \times 10 \times 10 = 67,600,000$$

Fundamental Counting Principle

Theorem 2.1. Multiplication Principle

With m elements $a_1, a_2, \ldots, a_m$ and n elements $b_1, b_2, \ldots, b_n$, it is possible to form $mn = m \times n$ pairs containing one element from each group.

- Example 2: How many CT license plates, consisting of 2 unique letters followed by 5 unique digits are possible?

$$26 \times 25 \times 10 \times 9 \times 8 \times 7 \times 6 = 19,656,000$$

Every time an object is chosen there is one less option for the next position.

Permutation

- When the arrangement of the objects is taken into account, we use the concept of permutation.
- A permutation gives the number of ways to order r objects out of n total objects where each object is different.
- Formula:

$$P_r^n = \frac{n!}{(n-r)!}$$

Permutation with Repetition

- If some of the objects of a permutation are repeated, the number of distinguishable permutations of all the objects is

$$P_r^n = \frac{n!}{q_1!q_2!q_3!\ldots},$$

where $q_1, q_2, q_3, \ldots$ are the number of times each object is repeated.

- Example: How many distinguishable ways are there to order the letters in the word TEETER?

The letter E is repeated 3x, hence, let $q_1 = 3$. The letter T is repeated 2x, hence, let $q_2 = 2$. No other letters are repeated. There are 6 total letters so $n = 6$. Therefore,

$$\begin{aligned} P_r^n &= \frac{n!}{q_1!q_2!} \\ P_6^6 &= \frac{6!}{3!2!} = 60. \end{aligned}$$

Combination

- When there is a need to group a limited number of items, we use the concept of combination.
- A combination gives the number of ways to group r objects out of n total objects without order where each object is different.
- Formula:

$$C_r^n = \frac{n!}{(n-r)!r!}$$

- Example 1: Five hundred boys, including Josh and Sam, entered a drawing for two football game tickets. What is the probability that the tickets were won by Josh and Sam?

We need to choose 2 students out of 500.
The total number of possible pairings is $C_2^{500} = \frac{500!}{(500-2)!2!} = \frac{500 \times 499}{2} = 124,750$.
The number of favorable outcome is 1.

$$P(\text{Josh and Sam won}) = \frac{1}{124,750}.$$

Combination

- When there is a need to group a limited number of items, we use the concept of combination.
- A combination gives the number of ways to group r objects out of n total objects without order where each object is different.
- Formula:

$$C_r^n = \frac{n!}{(n-r)!r!}$$

- Example 2: Suppose we keep dealing cards from an ordinary 52-card deck until the first jack appears. What is the probability that at least 10 cards go by before the first jack?

This is the probability that the first ten cards contain no Jack.

$$\begin{aligned} P(\text{no Jack in first 10 cards}) &= \frac{C_{10}^{48}}{C_{10}^{52}} \\ &= \frac{246}{595} = 0.4134. \end{aligned}$$

Conditional Probability

Conditional Probability: $P(A|B)$

- Will the occurrence of a previous event change the probability of occurrence of another event?
- As you obtain additional information, how should you update probabilities of events?
- We need the concept of *conditional probabilities* to answer this question.
- Denote by $P(A|B)$ the probability of event A occurring after it is assumed that event B has already happened.
  - It reads: "the probability of A given B".
- Formula:

$$P(A|B) = \frac{\text{probability of events A and B both occurring}}{\text{probability of event B occurring}} = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) > 0.$$

Conditional Probability: $P(A|B)$

Does this formula makes sense?

$$P(A|B) = \frac{\text{probability of events A and B both occurring}}{\text{probability of event B occurring}} = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) > 0.$$

Remember the formula for computing probabilities:

$$\text{probability} = \frac{\text{number of elements inside desired outcomes}}{\text{number of elements inside sample space}}$$

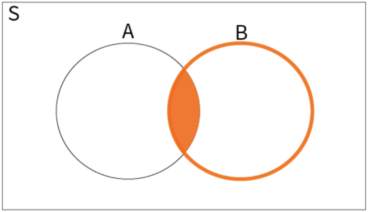


Figure: desired outcomes and sample space for conditional probabilities

Conditional Probability: $P(A|B)$

- Example 1: The probability that an automobile being filled with gasoline needs an oil change is 0.25; the probability that it also needs a new oil filter is 0.40; and the probability that both the oil and the filter need changing is 0.14.

- If the oil has to be changed, what is the probability that a new oil filter is needed?

$$P(\text{need oil filter change}|\text{need oil change})$$
$$= \frac{P(\text{need oil filter and oil change})}{P(\text{need oil change})}$$
$$= \frac{0.14}{0.25}$$
$$= 0.56$$

- If a new oil filter is needed, what is the probability that the oil has to be changed?

$$P(\text{need oil change}|\text{need oil filter change})$$
$$= \frac{P(\text{need oil filter and oil change})}{P(\text{need oil filter change})}$$
$$= \frac{0.14}{0.40}$$
$$= 0.35$$

Note how $P(A|B)$ is not equal to $P(B|A)$.

Independence

Independence

- Two events are said to be independent if knowing one occurs does not change the probability of the other occurring.

Definition 2.10

Two events A and B are said to be independent if any one of the following holds:

$$
\begin{aligned}
P(A|B) &= P(A) \\
P(B|A) &= P(B) \\
P(A \cap B) &= P(A)P(B)
\end{aligned}
$$

Otherwise, the events are said to be *dependent*.

Independent vs. Disjoint Events

Caution: Do not confuse independent events with disjoint or mutually exclusive events...

Concept	**Definition**	**Formulas**
disjoint events	A and B cannot occur at the same time.	$A \cap B = \emptyset$, $P(A\|B) = 0$, $P(B\|A) = 0$
independent events	B does not give any information about A.	$P(A\|B) = P(A)$, $P(B\|A) = P(B)$

- Example 1: I pick a random number from $\{1, 2, \ldots, 10\}$, and call it N. Suppose that all outcomes are equally likely. Let A be the event that N is less than 7, and let B be the event that N is an even number. Are A and B independent?

  Solution:

  - Given:
    1. all outcomes are equally likely:
       $P(N = 1) = P(N = 2) = P(N = 3) = P(N = 4) = P(N = 5) = P(N = 6) = P(N = 7) = P(N = 8) = P(N = 9)P(N = 10) = \frac{1}{10}$.
    2. A is the event that N is less than 7: $A = \{1, 2, 3, 4, 5, 6\}$.
    3. B is the event that N is an even number: $B = \{2, 4, 6, 8, 10\}$.
  - Question: Are A and B independent?
  - Strategy: Check if $P(A \cap B) = P(A)P(B)$. Independent if true.

    - ★ $A \cap B = \{2, 4, 6\}$.
    - ★ $P(A \cap B) = \frac{3}{10} = 0.3$.
    - ★ $P(A) = \frac{6}{10} = 0.6$.
    - ★ $P(B) = \frac{5}{10} = 0.5$.

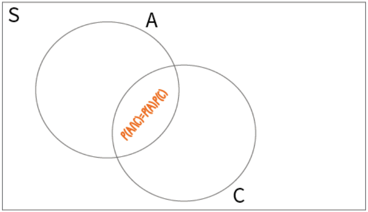
$$\longrightarrow \quad \begin{aligned} P(A \cap B) &\overset{?}{=} P(A)P(B) \\ 0.3 &\overset{?}{=} (0.6)(0.5) \\ 0.3 &\overset{\checkmark}{=} 0.3. \end{aligned}$$

- Example 2: For three events A, B, and C, we know that
  - A and C are independent,
  - B and C are independent,
  - A and B are disjoint, and
  - $P(A \cup C) = \frac{2}{3}, P(B \cup C) = \frac{3}{4}, P(A \cup B \cup C) = \frac{11}{12}$.

Find $P(A), P(B)$, and $P(C)$.

Solution:

- Given:
  1. A and C are independent.



Independence

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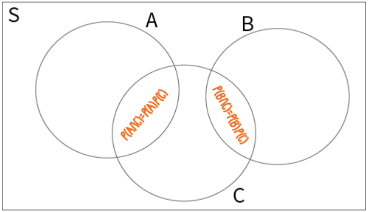
  - $P(A \cup C) = \frac{2}{3}, P(B \cup C) = \frac{3}{4}, P(A \cup B \cup C) = \frac{11}{12}$.

Find $P(A), P(B)$, and $P(C)$.

Solution:

- Given:

  2. B and C are independent. Also, A and B are disjoint.



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  Find $P(A), P(B)$, and $P(C)$.

  Solution:
  - Question: Solve for $P(A)$, $P(B)$, and $P(C)$.
  - Strategy:
    - ★ Use the additive law of probability.

$$P(A \cup C) = P(A) + P(C) - P(A \cap C) \quad \text{(Eq. 1)}$$
$$P(B \cup C) = P(B) + P(C) - P(B \cap C) \quad \text{(Eq. 2)}$$
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad \text{(Eq. 3)}$$

Independence

- Example 2: For three events A, B, and C, we know that
  - A and C are independent,
  - B and C are independent,
  - A and B are disjoint, and
  - $P(A \cup C) = \frac{2}{3}, P(B \cup C) = \frac{3}{4}, P(A \cup B \cup C) = \frac{11}{12}$.

  Find $P(A), P(B)$, and $P(C)$.

  Solution:
  - Question: Solve for $P(A)$, $P(B)$, and $P(C)$.
  - Strategy:
    - ★ Use the additive law of probability.

      $\frac{2}{3} = P(A) + P(C) - P(A \cap C)$ (Eq. 1)

      $\frac{3}{4} = P(B) + P(C) - P(B \cap C)$ (Eq. 2)

      $\frac{11}{12} = P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C)$ (Eq. 3)

      (since A & B are disjoint and A & B & C has no intersection)

Independence

- Example 2: For three events A, B, and C, we know that
  - A and C are independent,
  - B and C are independent,
  - A and B are disjoint, and
  - $P(A \cup C) = \frac{2}{3}, P(B \cup C) = \frac{3}{4}, P(A \cup B \cup C) = \frac{11}{12}$.

  Find $P(A), P(B)$, and $P(C)$.

  Solution:
  - Question: Solve for $P(A)$, $P(B)$, and $P(C)$.
  - Strategy:
    - ★ Use the additive law of probability.

      $\frac{2}{3} = P(A) + P(C) - P(A \cap C)$ (Eq. 1)

      $\frac{3}{4} = P(B) + P(C) - P(B \cap C)$ (Eq. 2)

      $\frac{11}{12} = P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C)$ (Eq. 3)
    - ★ Add Eqs 1 and 2.

      $\frac{17}{12} = P(A) + P(B) + 2P(C) - P(A \cap C) - P(B \cap C)$ (Eq. 1) + (Eq. 2)

      $\frac{11}{12} = P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C)$ (Eq. 3)

Independence

- Example 2: For three events A, B, and C, we know that
  - A and C are independent,
  - B and C are independent,
  - A and B are disjoint, and
  - $P(A \cup C) = \frac{2}{3}, P(B \cup C) = \frac{3}{4}, P(A \cup B \cup C) = \frac{11}{12}$.

  Find $P(A), P(B)$, and $P(C)$.

  Solution:
  - Question: Solve for $P(A)$, $P(B)$, and $P(C)$.
  - Strategy:
    - ★ Use the additive law of probability.

      $\frac{2}{3} = P(A) + P(C) - P(A \cap C)$ (Eq. 1)

      $\frac{3}{4} = P(B) + P(C) - P(B \cap C)$ (Eq. 2)

      $\frac{11}{12} = P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C)$ (Eq. 3)
    - ★ Add Eqs 1 and 2.

      $\frac{17}{12} = P(A) + P(B) + 2P(C) - P(A \cap C) - P(B \cap C)$ (Eq. 1 + Eq. 2)

      $\frac{11}{12} = P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C)$ (Eq. 3)
    - ★ Subtract Eq. 3 from (Eq. 1 + Eq. 2).

      $\frac{6}{12} = P(C)$ (Eq. 1 + Eq. 2) - (Eq. 3)

      $\frac{1}{2} = P(C)$.

Independence

- Example 2: For three events A, B, and C, we know that
  - A and C are independent,
  - B and C are independent,
  - A and B are disjoint, and
  - $P(A \cup C) = \frac{2}{3}, P(B \cup C) = \frac{3}{4}, P(A \cup B \cup C) = \frac{11}{12}$.

  Find $P(A), P(B)$, and $P(C)$.

  Solution:
  - Question: Solve for $P(A)$, $P(B)$, and $P(C)$.
  - Answers:

★ $P(C) = 0.5$

★ Find values of $P(A)$.

$\frac{2}{3} = P(A) + P(C) - P(A \cap C)$ (Eq. 1)
$\frac{2}{3} = P(A) + \frac{1}{2} - P(A \cap C)$
$\frac{2}{3} = P(A) + \frac{1}{2} - P(A)P(C)$ (A & C independent)
$\frac{2}{3} - \frac{1}{2} = P(A) - \frac{1}{2}P(A)$
$\frac{1}{6} = \frac{1}{2}P(A)$
$\frac{1}{3} = P(A)$.

★ Find values of $P(B)$.

$\frac{3}{4} = P(B) + P(C) - P(B \cap C)$ (Eq. 2)
$\frac{3}{4} = P(B) + \frac{1}{2} - P(B \cap C)$
$\frac{3}{4} = P(B) + \frac{1}{2} - P(B)P(C)$ (B & C independent)
$\frac{3}{4} - \frac{1}{2} = P(B) - \frac{1}{2}P(B)$
$\frac{1}{4} = \frac{1}{2}P(B)$
$\frac{1}{2} = P(B)$.

Questions?

Homework Exercises: 2.15, 2.17, 2.19, 2.21, 2.23, 2.75, 2.77

Solutions will be discussed this Friday by the TA.