STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 4: Bayes' Rule; Discrete Random Variables

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Outline

1. Quiz Solutions
2. Previously...
   - Probabilities
3. Random Variables
   - Motivation
   - Definition
   - Discrete Random Variables
   - The Probability Distribution

Quiz Solutions

Problem 3

In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

UNCLEAR SOLUTIONS:

- $\frac{\text{4 freshman boys}}{\text{10 freshmen}} = \frac{\text{6 sophomore boys}}{6+x \text{ sophomores}}$
  What is this supposed to be? Ratio? Fraction? Probability?
- If that was supposed to be computing for the probability, then what you actually wrote is:
  $\frac{P(F \cap B)}{P(F)} = \frac{P(S \cap B)}{P(S)} \Rightarrow P(B|F) = P(B|S)$.
- To show independence, choose one:

$$\begin{aligned} P(A|B) &= P(A) \\ P(B|A) &= P(B) \\ P(A \cap B) &= P(A)P(B) \end{aligned}$$

Problem 3

In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

UNCLEAR SOLUTIONS:

- $\frac{\text{4 freshman boys}}{\text{10 freshmen}} = \frac{\text{6 freshman girls}}{6+x \text{ girls}}$
  What is this supposed to be? Ratio? Fraction? Probability?
- If that was supposed to be computing for the probability, then what you actually wrote is:
  $\frac{P(F \cap B)}{P(F)} = \frac{P(F \cap G)}{P(G)} \Rightarrow P(B|F) = P(F|G)$.
- To show independence, choose one:

$$\begin{aligned} P(A|B) &= P(A) \\ P(B|A) &= P(B) \\ P(A \cap B) &= P(A)P(B) \end{aligned}$$

Previously...

Probabilities

- Probability: Suppose A is an event in sample space S.

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$

- Conditional Probability: As you obtain additional information, how should you update probabilities of events?

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) > 0.$$

- Total Probability: Assume that $\{B_1, B_2, \ldots, B_k\}$ is a partition of S such that $P(B_i) > 0$ for $i = 1, 2, \ldots, k$.

$$P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i).$$

- Bayes' Rule: What is the probability of a cause, given an effect?

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}.$$

Example 1

Suppose that if I take Bus 1, then I am late with probability 0.1. If I take Bus 2, I am late with probability 0.2. The probability that I take Bus 1 is 0.4, and the probability that I take Bus 2 is 0.6. What is the probability that I am late?

Solution:

- Question: What is $P(\text{late})$?
- Given: $P(\text{late}|\text{Bus 1}) = 0.1$, $P(\text{late}|\text{Bus 2}) = 0.2$, $P(\text{Bus 1}) = 0.4$, and $P(\text{Bus 2}) = 0.6$.
- Strategy: Total probability formula

$$
\begin{aligned}
P(A) &= \sum_{i=1}^{k} P(A|B_i)P(B_i) \\
P(\text{late}) &= P(\text{late}|\text{Bus 1})P(\text{Bus 1}) + P(\text{late}|\text{Bus 2})P(\text{Bus 2}) \\
&= (0.1)(0.4) + (0.2)(0.6) = \mathbf{0.16}.
\end{aligned}
$$

Example 2

Suppose that if I take Bus 1, then I am late with probability 0.1. If I take Bus 2, I am late with probability 0.2. The probability that I take Bus 1 is 0.4, and the probability that I take Bus 2 is 0.6. Suppose I was late. What was the probability that I took Bus 2?

Solution:

- Question: What is $P(\text{Bus 2}|\text{late})$?
- Given: $P(\text{late}|\text{Bus 1}) = 0.1$, $P(\text{late}|\text{Bus 2}) = 0.2$, $P(\text{Bus 1}) = 0.4$, and $P(\text{Bus 2}) = 0.6$.
- Strategy: Bayes' Rule *(prob. of cause given effect)*

$$
\begin{aligned}
P(B_j|A) &= \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)} \\
P(\text{Bus 2}|\text{late}) &= \frac{P(\text{late}|\text{Bus 2})P(\text{Bus 2})}{P(\text{late}|\text{Bus 1})P(\text{Bus 1}) + P(\text{late}|\text{Bus 2})P(\text{Bus 2})} \\
&= \frac{(0.2)(0.6)}{(0.1)(0.4) + (0.2)(0.6)} = \mathbf{0.75}.
\end{aligned}
$$

Example 2

Suppose that if I take Bus 1, then I am late with probability 0.1. If I take Bus 2, I am late with probability 0.2. The probability that I take Bus 1 is 0.4, and the probability that I take Bus 2 is 0.6. Suppose I was late. What was the probability that I took Bus 2?

Another Solution:

- Question: What is $P(\text{Bus 2}|\text{late})$?
- Given: $P(\text{late}|\text{Bus 1}) = 0.1$, $P(\text{late}|\text{Bus 2}) = 0.2$, $P(\text{Bus 1}) = 0.4$, and $P(\text{Bus 2}) = 0.6$.
- Strategy: Bayes' Rule *(prob. of cause given effect)*

$$
\begin{aligned}
P(B_j|A) &= \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)} = \frac{P(A|B_j)P(B_j)}{P(A)} \\
P(\text{Bus 2}|\text{late}) &= \frac{P(\text{late}|\text{Bus 2})P(\text{Bus 2})}{P(\text{late}) \quad \text{(Answer to Ex. 1)}} \\
&= \frac{(0.2)(0.6)}{0.16} = \mathbf{0.75}.
\end{aligned}
$$

Random Variables

Random Variables: Motivation

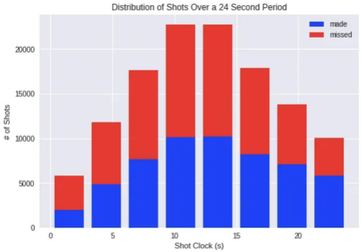

Figure: No. of shots by all NBA players in a 24 sec. shot clock period in the 2014–2015 season.

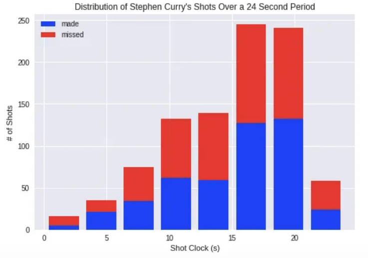

Figure: No. of shots by Steph Curry in a 24 sec. shot clock period in the 2014–2015 season.

Random variables enable us to...

- *paint the bigger of picture of any probabilistic experiment.*
- *answer more interesting questions such as:*
  - *Are quick shots better than shots with the shot clock expiring?*
  - *What makes Steph Curry phenomenal?*

https://ahtan-18882.medium.com/beating-the-nba-shot-clock-and-more-exploratory-data-analysis-project-6e2be3649426

Random Variables: Motivation

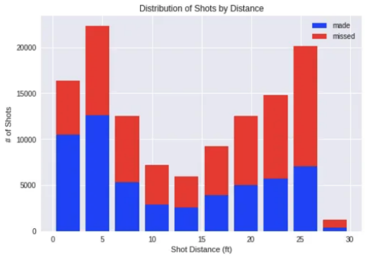

Figure: No. of shots by all NBA players by distance in the 2014–2015 season.

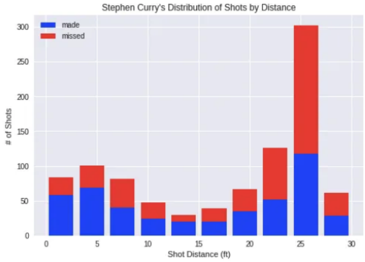

Figure: No. of shots by Steph Curry by distance in the 2014–2015 season.

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Random Variables: Motivation

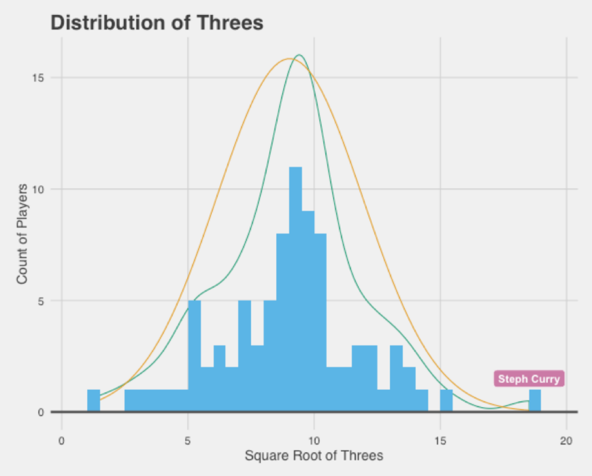


Figure: No. of 3-point shots made in the 2015–2016 NBA season. Steph Curry has made 343 threes.

Source: https://jamesmccammon.com/2016/03/26/steph-curry-is-awesome/

Random Variables: Motivation

Figure: Netflix adjusted closing price.

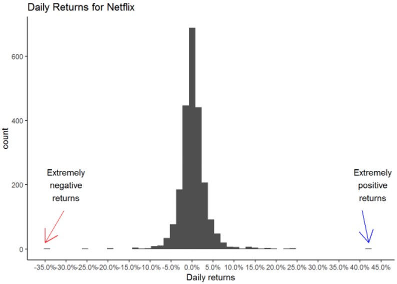


Figure: Netflix daily returns.

Source: https://www.codingfinance.com/post/2018-04-03-calc-returns/

Random Variables: Definition

Definition 2.12

A *random variable* is a real-valued function for which the domain is a sample space.

- A random variable is NOT random.
- A random variable is NOT a variable.
- A random variable is a function that maps outcomes into real numbers.
- One can think of a random variable as a numerical summary of what happened in a probability experiment.
- A random variable is commonly denoted by a capital letter. The letters X, Y, and Z are more commonly used.

Random Variables: As a Mapping from S to $\mathbb{R}$

A random variable is a <u>function</u> that maps outcomes into real numbers...

- Function: a mapping of each input to exactly one output
- Example 1: Three fair coins are flipped. Let the random variable X be the number of heads obtained. What are the possible values of X?

X is the number of heads

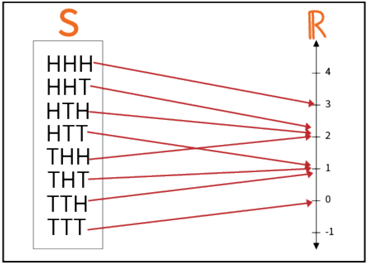


Figure: X maps the outcomes in S to the set $\mathbb{R}$ of real numbers.

Answer: $X = 0, 1, 2, 3.$

Random Variables: As a Mapping from S to $\mathbb{R}$

A random variable is a <u>function</u> that maps outcomes into real numbers...

- Example 2: Three fair coins are flipped. Let the random variable Y be the absolute difference between the number of heads and the number of tails. What are the possible values of Y?

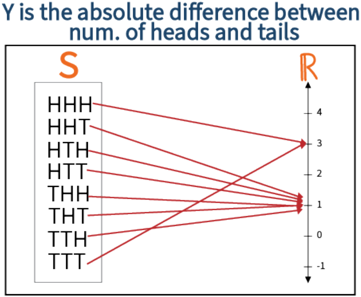


Figure: Y maps the outcomes in S to the set $\mathbb{R}$ of real numbers.

Answer: $Y = 1, 3$.

A random variable is a <u>function</u> that maps outcomes into real numbers...

- Example 3: Let X_1 be the random variable denoting the roll of the first die. Let X_2 be the random variable denoting the roll of the second die. Let $Z = \max(X_1, X_2)$. What are the possible values of Z?

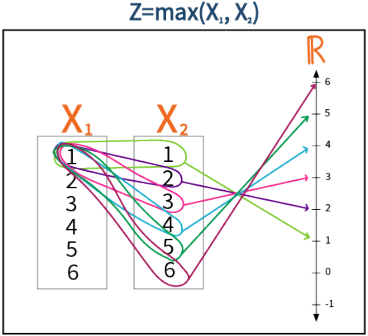


Figure: Z maps the outcomes in S to the set $\mathbb{R}$ of real numbers.

Answer: $Z = 1, 2, 3, 4, 5, 6.$

Types of Random Variables

Random variables can classified into two types based on the possible values that they can take.

- Discrete random variables: random variables whose values take only a finite or countably infinite number of possible values.

  Example: Let X be the roll on the die. There are only a finite number of possible values for X, namely, 1, 2, 3, 4, 5, and 6.

- Continuous random variables: are random variables that can take on an infinite number of possible values.

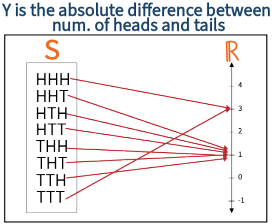
  Example: Stock prices, temperature, wind speed

Discrete Random Variables

Definition 3.1

A *random variable* Y is said to be *discrete* if it can assume only a finite or countably infinite number of distinct values.

- Each possible value of a discrete random variable contains a certain probability.



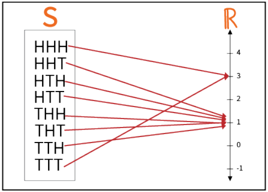
For example: $P(Y = 3) =?$ and $P(Y = 1) =?$

- probability distribution: collection of probabilities of the discrete random variable.

The Probability Distribution of Discrete Random Variables

- Some notations:
  - random variables: uppercase letter.
    Example: Let Y be the absolute difference between the number of heads and the number of tails.
  - particular value of the random variable (result): lowercase letter.
    Example: $y = 3$, $y = 1$.
- Goal: Compute $P(Y = y)$.
  Read as: *Probability that Y takes on the value y.*

Y is the absolute difference between num. of heads and tails



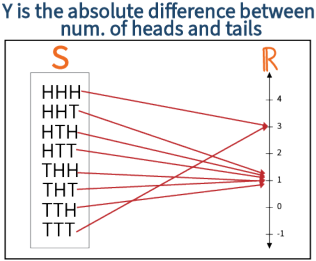
Definition 3.2

The probability that Y takes on the value y, denoted $P(Y = y)$, is defined as the sum of the probabilities of all sample points in S that are assigned the value y.

Note: $P(Y = y)$ is also often written as $p(y)$.

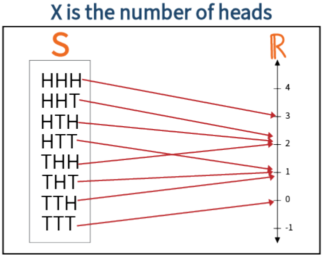
The Probability Distribution of Discrete Random Variables

- Example 1:



- Goal: Compute $p(y)$ or $P(Y = y)$.
  1. What is y? Answer: $y = 1, 3$.
  2. What is $p(1)$ or $P(Y = 1)$?
     Answer:
     $p(1) = P(Y = 1) = P(\{HHT, HTH, HTT, THH, THT, TTH\}) = \frac{6}{8} = \frac{3}{4}$.
  3. What is $p(3)$ or $P(Y = 3)$?
     Answer: $p(3) = P(Y = 3) = P(\{HHH, TTT\}) = \frac{2}{8} = \frac{1}{4}$.
  4. What is $p(2)$ or $P(Y = 2)$?
     Answer: $p(2) = P(Y = 2) = 0$.

- Example 2:



- Goal: Compute $p(x)$ or $P(X = x)$.
  1. What is x? Answer: $x = 0, 1, 2, 3$.
  2. What is $p(0)$ or $P(X = 0)$?
     Answer: $p(0) = P(X = 0) = P(\{TTT\}) = \frac{1}{8}$.
  3. What is $p(1)$ or $P(X = 1)$?
     Answer: $p(1) = P(X = 1) = P(\{HTT, THT, TTH\}) = \frac{3}{8}$.
  4. What is $p(2)$ or $P(X = 2)$?
     Answer: $p(2) = P(X = 2) = P(\{HHT, HTH, THH\}) = \frac{3}{8}$.
  5. What is $p(3)$ or $P(X = 3)$?
     Answer: $p(3) = P(X = 3) = P(\{HHH\}) = \frac{1}{8}$.

The Probability Distribution of Discrete Random Variables

Definition 3.3

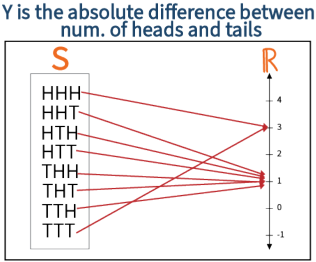
The probability distribution for a discrete variable Y can be represented by a formula, a table, or a graph that provides $p(y) = P(Y = y)$ for all y.

Some Remarks:

- $p(y)$ is a function that assigns probabilities to each value y of the random variable Y.
- $p(y)$ is often referred to as the probability function for Y.

The Probability Distribution of Discrete Random Variables

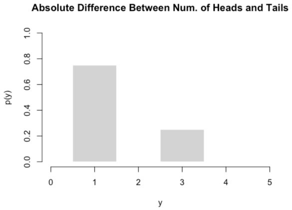
- Example 1:



- Table of the probability distribution:

y	$p(y)$
1	$1/4$
3	$3/4$

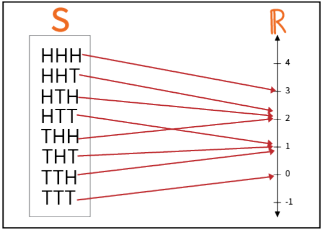
- Histogram of the probability distribution:



The Probability Distribution of Discrete Random Variables

- Example 2:

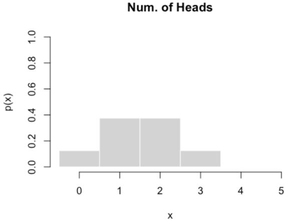
X is the number of heads



- Table of the probability distribution:

x	$p(x)$
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$

- Histogram of the probability distribution:



The Probability Distribution of Discrete Random Variables

Theorem 3.1

For any discrete probability distribution, the following must be true:

1. $0 \leq p(y) \leq 1$, for all y.
2. $\sum_y p(y) = 1$, where the summation is over all values of y with nonzero probability.

A basketball player takes 4 independent free throws with a probability of 0.7 of getting a basket on each shot. Let $Y =$ the number of baskets he gets. Write out the full probability distribution for Y.

Answer:

y	$p(y)$
0	$C_0^4(1-0.7)(1-0.7)(1-0.7)(1-0.7) = 0.0081$
1	$C_1^4(0.7)(1-0.7)(1-0.7)(1-0.7) = 0.0756$
2	$C_2^4(0.7)(0.7)(1-0.7)(1-0.7) = 0.2646$
3	$C_3^4(0.7)(0.7)(0.7)(1-0.7) = 0.4116$
4	$C_4^4(0.7)(0.7)(0.7)(0.7) = 0.2401$

Questions?

Homework Exercises: 3.7, 3.19, 3.27, 3.31, 3.33

Solutions will be discussed this Friday by the TA.