STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 6: Special Discrete Distributions: Bernoulli, Binomial, Geometric

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   - Variance of Discrete Random Variables
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4. Special Discrete Distributions
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   - Bernoulli Distribution
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Quiz 1 Solutions (Again...)

Problem 3

In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

MAJORITY OF YOUR SOLUTIONS:

	freshmen (F)	sophomores (S)	total
boys (B)	4	6	10
girls (G)	6	x	6 + x
total	10	6 + x	

- If sex and class are independent, then

$$\begin{aligned} P(B|F) &= P(B|S) \\ \frac{4}{10} &= \frac{6}{6+x} \\ \Rightarrow x &= 9. \end{aligned}$$

- Skipped the first two steps: If sex and class are independent, then
  - $P(B|F) = P(B)$
  - $P(B|S) = P(B)$

  Therefore, $P(B|F) = P(B|S)$. Proceed with your solution. I'll give partial points...

Previously...

Expected Value of Discrete Random Variables

- Expected value of Y: is a measure of central tendency

$$\mu = E(Y) = \sum_y y p(y)$$

- Expected value of functions of Y: Let $g(Y)$ be a real-valued function of Y.

$$E\{g(Y)\} = \sum_y g(y)p(y)$$

- Expected value of a constant: Let c be a constant. Then $E(c) = c$.
- Expected value of a scaled Y: Let $g(Y)$ be a function of Y, and c be a constant.

$$E\{cg(Y)\} = cE\{g(Y)\}$$

- Expected value of a sum of random variables: Let $g_1(Y), g_2(Y), \ldots, g_k(Y)$ be k functions of Y.

$$E\{g_1(Y) + g_2(Y) + \ldots + g_k(Y)\} = E\{g_1(Y)\} + E\{g_2(Y)\} + \ldots + E\{g_k(Y)\}$$

Variance of Discrete Random Variables

- Variance of Y: is a measure of the dispersion or scatter of the values of the random variable about the mean μ.

$$\sigma^2 = V(Y) = E\{(Y - \mu)^2\}$$

- More useful formula to compute the variance:

$$\sigma^2 = V(Y) = E(Y^2) - \mu^2$$

Note: The formula above can also be written as $\sigma^2 = V(Y) = E(Y^2) - \{E(Y)\}^2$.

- Standard deviation of Y:

$$\sigma = \sqrt{V(Y)}$$

Variance of Discrete Random Variables

- Variance of a scaled random variable: Let c be a constant. Then $V(cY) = c^2 V(Y)$.

  Note: Unlike the expected value, variance is not a linear function.

  Proof:

$$\begin{aligned} V(cY) &= E\{(cY)^2\} - \{E(cY)\}^2 && \text{defn of variance: } V(Y) = E(Y^2) - \{E(Y)\}^2 \\ &= E(c^2Y^2) - \{cE(Y)\}^2 && \text{expected value of scaled random variables} \\ &= c^2E(Y^2) - c^2\{E(Y)\}^2 && \text{expected value of scaled random variables} \\ &= c^2[E(Y^2) - \{E(Y)\}^2] && \text{factor out } c^2 \\ &= c^2V(Y). && \text{defn of variance} \end{aligned}$$

- Variance of a constant: Let c be a constant. Then $V(c) = 0$.

  Proof:

$$\begin{aligned} V(c) &= E(c^2) - \{E(c)\}^2 && \text{defn of variance} \\ &= c^2 - (c)^2 && \text{expected value of a constant is itself} \\ &= 0. \end{aligned}$$

Variance of Discrete Random Variables

- Variance of $Y = aX + b$: Let X be a random variable and a and b be constants. Then $V(Y) = a^2 V(X)$.

  Proof:

$$
\begin{aligned}
V(Y) &= E\{(aX+b)^2\} - \{E(aX+b)\}^2 \\
&\qquad \text{defn of variance: } V(Y) = E(Y^2) - \{E(Y)\}^2 \\
&= E\{(aX+b)^2\} - \{aE(X) + E(b)\}^2 \\
&\qquad \text{expected value of functions and scaled random variables} \\
&= E\{(aX+b)^2\} - \{aE(X) + b\}^2 \quad \text{expected value of constant is itself} \\
&= E\{(aX+b)^2\} - [a^2\{E(X)\}^2 + 2abE(X) + b^2] \quad \text{squaring} \\
&= E\{(aX+b)^2\} - a^2\{E(X)\}^2 - 2abE(X) - b^2 \quad \text{distribute neg. sign} \\
&= E(a^2X^2 + 2abX + b^2) - a^2\{E(X)\}^2 - 2abE(X) - b^2 \quad \text{squaring} \\
&= a^2E(X^2) + 2abE(X) + b^2 - a^2\{E(X)\}^2 - 2abE(X) - b^2 \\
&\qquad \text{expected value of functions and scaled random variables} \\
&= a^2E(X^2) - a^2\{E(X)\}^2 \quad \text{cancel out terms} \\
&= a^2[E(X^2) - \{E(X)\}^2] \quad \text{factor out } a^2 \\
&= a^2V(X). \quad \text{defn of variance}
\end{aligned}
$$

The Probability Mass Function

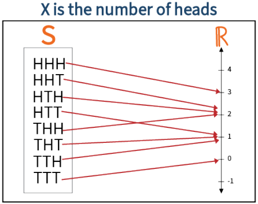
The Probability Mass Function

Recall in Lecture 4...

Definition 3.3

The probability distribution for a discrete variable Y can be represented by a formula, a table, or a graph that provides $p(y) = P(Y = y)$ for all y.

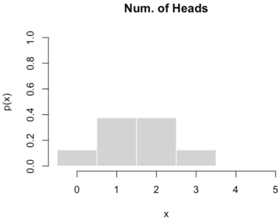
- Example:



- Table of the probability distribution:

x	$p(x)$
0	1/8
1	3/8
2	3/8
3	1/8

- Histogram of the probability distribution:



We have not yet discussed how to represent the probability distribution of Y by a formula.

The Probability Mass Function

- A discrete probability distribution can be represented by a formula, commonly termed as probability mass function (pmf).
- A pmf is a function that gives the probability of a discrete random variable Y being exactly equal to some value, y.
- The pmf depends on certain constants which are more commonly called parameters.
- The parameters dictate the shape and location of the graph of the probability distribution on the real line (histogram).
- The pmf will be an important concept in the discussion of special discrete distributions later...

The Probability Mass Function

Recall in Lecture 4...

A basketball player takes 4 independent free throws with a probability of 0.7 of getting a basket on each shot. Let $Y =$ the number of baskets he gets. Write out the full probability distribution for Y.

Answer:

y	$p(y)$
0	$C_0^4(1-0.7)(1-0.7)(1-0.7)(1-0.7) = 0.0081$
1	$C_1^4(0.7)(1-0.7)(1-0.7)(1-0.7) = 0.0756$
2	$C_2^4(0.7)(0.7)(1-0.7)(1-0.7) = 0.2646$
3	$C_3^4(0.7)(0.7)(0.7)(1-0.7) = 0.4116$
4	$C_4^4(0.7)(0.7)(0.7)(0.7) = 0.2401$

We can summarize the probabilities above using the formula or pmf:

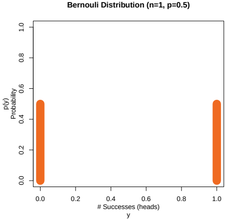
$$p(y) = C_y^4 p^y (1-p)^{4-y},$$

where $y = 0, 1, 2, 3, 4$ and $p = 0.7$.

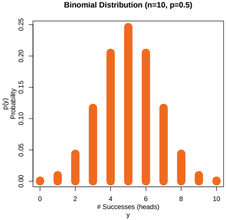
Special Discrete Distributions

Motivation

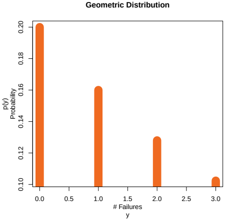
Depending on what the random variable represents, its probability distribution can be classified into one of these distributions...



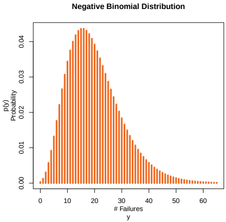
success or failure?



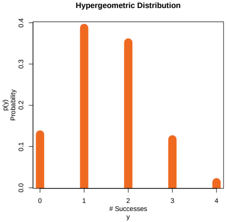
num of successes in n attempts?



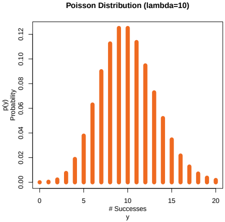
num of failures before first success?



num of failures before r successes?



num of successes in n attempts (w/o replacement)?



num of times an event occurs in a specified time interval?

Bernoulli Distribution

Definition: Bernoulli Distribution

A random variable Y is said to have a *Bernoulli distribution* with success probability p if and only if

$$p(y) = p^y(1-p)^{1-y},$$

where $y = 0, 1$ and $0 \leq p \leq 1$.

- the probability distribution of a random variable taking on only two values, 0 ("failure") with probability $1-p$ and 1 ("success") with probability p
- Appropriate for describing events having exactly two (binary) outcomes
  Examples: a team will win a championship or not, a student will pass or fail an exam, and a rolled dice will either show a 6 or any other number.
- We call $p(y) = p^y(1-p)^{1-y}$ as the pmf of a Bernoulli random variable.
- Notation: $Y \sim \text{Bern}(p)$, read as: "Y is a Bernoulli random variable with probability p."

Bernoulli Distribution

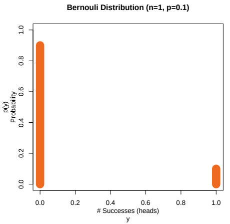
- The expected value of a Bernoulli random variable Y is:

$$E(Y) = 0(1-p) + 1(p) = p.$$

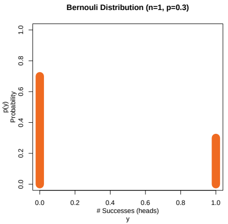
- The variance of a Bernoulli random variable Y is:

$$V(Y) = E(Y^2) - \{E(Y)\}^2 = 0^2(1-p) + 1^2 p - p^2 = p(1-p).$$

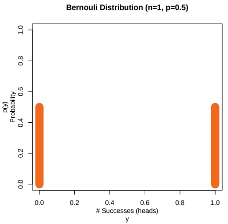
Bernoulli Distribution



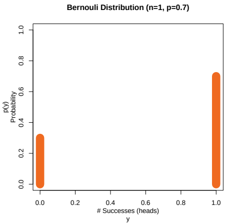
$E(Y) = p = 0.1, V(Y) = 0.09$



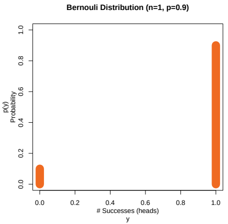
$E(Y) = p = 0.3, V(Y) = 0.21$



$E(Y) = p = 0.5, V(Y) = 0.25$



$E(Y) = p = 0.7, V(Y) = 0.21$



$E(Y) = p = 0.9, V(Y) = 0.09$

Bernoulli Distribution

Example:

The standard deviation of a Bernoulli random variable Y is $\frac{2}{5}$.

ⓐ Calculate the variance of Y.

ⓑ Calculate the expected value of Y.

Solution:

ⓐ $V(Y) = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$.

ⓑ The expected value of a Bernoulli random variable Y is $E(Y) = p$. This means we need to solve for p. To solve for p, we can use the variance above. Also, recall that the variance of a Bernoulli random variable Y is $V(Y) = p(1 - p)$. Thus,
$V(Y) = p(1 - p) = \frac{4}{25}$
$\Rightarrow p - p^2 = \frac{4}{25}$
$\Rightarrow p^2 - p + \frac{4}{25} = 0$
$\Rightarrow \left(p - \frac{1}{5}\right)\left(p - \frac{4}{5}\right) = 0$
$\Rightarrow p = \frac{1}{5}$ or $p = \frac{4}{5}$
$\Rightarrow E(Y) = \frac{1}{5}$ or $E(Y) = \frac{4}{5}$.

Binomial Distribution

Definition 3.7: Binomial Distribution

A random variable Y is said to have a *binomial distribution* based on n trials with success probability p if and only if

$$p(y) = \binom{n}{y} p^y (1-p)^{n-y},$$

where $\binom{n}{y} = \frac{n!}{y!(n-y)!}$, $y = 0, 1, 2, \ldots, n$ and $0 \le p \le 1$.

- the probability distribution of a random variable counting the number of successes in n independent trials, such that the probability of success at each trial is p
- We call $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$ as the pmf of a binomial random variable.
- Notation: $Y \sim \mathrm{B}(n, p)$, read as: "Y is a binomial random variable with n trials and probability of success p."
- The Bernoulli distribution is a special case of the binomial distribution wherein $n = 1$.

Binomial Distribution

Theorem 3.7

Let Y be a binomial random variable based on n trials and success probability p. Then

$$\mu = E(Y) = np \quad \text{and} \quad \sigma^2 = V(Y) = np(1-p).$$

Proof:

$$\begin{aligned} E(Y) &= \sum_y y p(y) \quad \text{defn of expected value} \\ &= \sum_{y=0}^{n} y \binom{n}{y} p^y (1-p)^{n-y} \quad \text{pmf of binomial distribution} \\ &= \sum_{y=1}^{n} y \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y} \end{aligned}$$

start index at $y = 1$ since $y = 0$ does not contribute to the summation

(continued next slide...)

Binomial Distribution

Proof:

$$
\begin{aligned}
E(Y) &= \sum_{y=1}^{n} y \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y} \\
&= \sum_{y=1}^{n} \frac{n!}{(y-1)!(n-y)!} p^y (1-p)^{n-y} \qquad \text{cancel out } y \text{ in num. \& denom.} \\
&= np \sum_{y=1}^{n} \frac{(n-1)!}{(y-1)!(n-y)!} p^{y-1} (1-p)^{n-y} \qquad \text{factor out } np \\
&= np \sum_{z=0}^{n-1} \frac{(n-1)!}{z!(n-1-z)!} p^z (1-p)^{n-1-z} \qquad \text{Let } z = y-1. \\
&= np \sum_{z=0}^{n-1} \binom{n-1}{z} p^z (1-p)^{n-1-z} \\
&= np \sum_{z} p(z) \qquad \text{binomial pmf with } n-1 \text{ trials \& } z \text{ successes: } p(y) = \binom{n}{y} p^y (1-p)^{n-y} \\
&= np. \qquad \text{probabilities sum to 1}
\end{aligned}
$$

□

Binomial Distribution

Proof: To compute the variance, we need to solve for $E(Y^2)$. However, the techniques we used earlier when solving for the expected value are not helpful for $E(Y^2)$ since we will get $E(Y^2) = \sum_{y=0}^{n} y^2 \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}$, wherein y^2 is not a factor of $y!$.

Instead, note that $E\{Y(Y-1)\} = E(Y^2 - Y) = E(Y^2) - E(Y)$.

Therefore, $E(Y^2) = E\{Y(Y-1)\} + E(Y)$.

Hence, we solve first $E\{Y(Y-1)\}$.

$$
\begin{aligned}
E\{Y(Y-1)\} &= \sum_{y} y(y-1)p(y) \quad \text{defn of expected value} \\
&= \sum_{y=0}^{n} y(y-1) \binom{n}{y} p^y (1-p)^{n-y} \quad \text{pmf of binomial distribution} \\
&= \sum_{y=2}^{n} y(y-1) \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}
\end{aligned}
$$

start index at $y = 2$ since $y = 0, 1$ don't contribute to the summation

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Binomial Distribution

Proof:

$$
\begin{aligned}
E\{Y(Y-1)\} &= \sum_{y=2}^{n} y(y-1)\frac{n!}{y!(n-y)!}p^y(1-p)^{n-y} \\
&= \sum_{y=2}^{n} \frac{n!}{(y-2)!(n-y)!}p^y(1-p)^{n-y} \quad \text{cancel out } y(y-1) \text{ in num. \& denom.} \\
&= n(n-1)p^2 \sum_{y=2}^{n} \frac{(n-2)!}{(y-2)!(n-y)!}p^{y-2}(1-p)^{n-y} \quad \text{factor out } n(n-1)p^2 \\
&= n(n-1)p^2 \sum_{z=0}^{n-2} \frac{(n-2)!}{z!(n-2-z)!}p^z(1-p)^{n-2-z} \quad \text{Let } z=y-2. \\
&= n(n-1)p^2 \sum_{z=0}^{n-2} \binom{n-2}{z} p^z(1-p)^{n-2-z} \\
&= n(n-1)p^2 \sum_{z} p(z) \quad \text{binomial pmf with } n-2 \text{ trials \& } z \text{ successes:}
\end{aligned}
$$

$$p(y) = \binom{n}{y} p^y(1-p)^{n-y}$$

$$= n(n-1)p^2. \quad \text{probabilities sum to 1}$$

Binomial Distribution

Proof:

We have $E\{Y(Y-1)\} = n(n-1)p^2$.

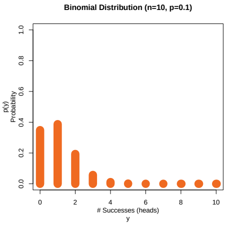
But we need $E(Y^2)$ which is:

$$\begin{aligned} E(Y^2) &= E\{Y(Y-1)\} + E(Y) \\ &= n(n-1)p^2 + np. \end{aligned}$$

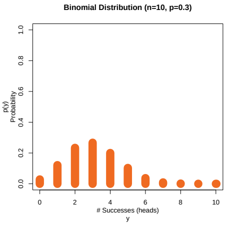
We can now solve for $V(Y)$ as follows:

$$\begin{aligned} V(Y) &= E(Y^2) - \{E(Y)\}^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= n^2p^2 - np^2 + np - n^2p^2 \\ &= -np^2 + np \\ &= np(-p+1) = np(1-p) \end{aligned}$$

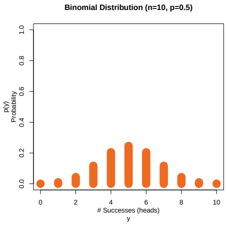
□



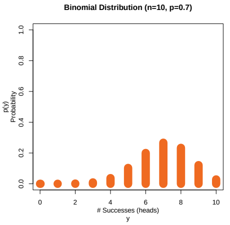
$E(Y) = 1, V(Y) = 0.9$



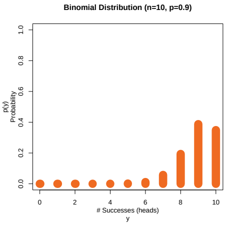
$E(Y) = 3, V(Y) = 2.1$



$E(Y) = 5, V(Y) = 2.5$



$E(Y) = 7, V(Y) = 2.1$



$E(Y) = 9, V(Y) = 0.9$

Binomial Distribution

Example 1:

The manufacturer of a low-calorie dairy drink wishes to compare the taste appeal of a new formula (formula B) with that of the standard formula (formula A). Each of four judges is given three glasses in random order, two containing formula A and the other containing formula B. Each judge is asked to state which glass he or she most enjoyed. Suppose that the two formulas are equally attractive. Let Y be the number of judges stating a preference for the new formula.

a. Find the probability (mass) function for Y.
b. What is the probability that at least three of the four judges state a preference for the new formula?
c. Find the expected value of Y.
d. Find the variance of Y.

Solution:

Note that Y is binomial with $n = 4$ and $p = 1/3 = P(\text{judge chooses formula } B)$.

a. $p(y) = \binom{4}{y} \left(\frac{1}{3}\right)^y \left(1 - \frac{1}{3}\right)^{4-y}$, $y = 0, 1, 2, 3, 4$.

pmf of binom: $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$

Binomial Distribution

Example 1:

The manufacturer of a low-calorie dairy drink wishes to compare the taste appeal of a new formula (formula B) with that of the standard formula (formula A). Each of four judges is given three glasses in random order, two containing formula A and the other containing formula B. Each judge is asked to state which glass he or she most enjoyed. Suppose that the two formulas are equally attractive. Let Y be the number of judges stating a preference for the new formula.

a. Find the probability (mass) function for Y.
b. What is the probability that at least three of the four judges state a preference for the new formula?
c. Find the expected value of Y.
d. Find the variance of Y.

Solution:

Note from a), the pmf of Y is $p(y) = \binom{4}{y} \left(\frac{1}{3}\right)^{y} \left(1 - \frac{1}{3}\right)^{4-y}$.

b. $P(Y \geq 3) = p(3) + p(4) = \binom{4}{3} \left(\frac{1}{3}\right)^{3} \left(1 - \frac{1}{3}\right)^{4-3} + \binom{4}{4} \left(\frac{1}{3}\right)^{4} \left(1 - \frac{1}{3}\right)^{4-4} = \frac{8}{81} + \frac{1}{81} = \frac{9}{81} = \frac{1}{9}$.

Example 1:

The manufacturer of a low-calorie dairy drink wishes to compare the taste appeal of a new formula (formula B) with that of the standard formula (formula A). Each of four judges is given three glasses in random order, two containing formula A and the other containing formula B. Each judge is asked to state which glass he or she most enjoyed. Suppose that the two formulas are equally attractive. Let Y be the number of judges stating a preference for the new formula.

a. Find the probability (mass) function for Y.
b. What is the probability that at least three of the four judges state a preference for the new formula?
c. Find the expected value of Y.
d. Find the variance of Y.

Solution:

Note that the expected value of a binomial r.v. is np and Y is a binomial r.v. with $n = 4$ and $p = 1/3 = P(\text{judge chooses formula } B)$.

c. $E(Y) = np = 4\left(\frac{1}{3}\right)$.

Binomial Distribution

Example 1:

The manufacturer of a low-calorie dairy drink wishes to compare the taste appeal of a new formula (formula B) with that of the standard formula (formula A). Each of four judges is given three glasses in random order, two containing formula A and the other containing formula B. Each judge is asked to state which glass he or she most enjoyed. Suppose that the two formulas are equally attractive. Let Y be the number of judges stating a preference for the new formula.

a. Find the probability (mass) function for Y.
b. What is the probability that at least three of the four judges state a preference for the new formula?
c. Find the expected value of Y.
d. Find the variance of Y.

Solution:

Note that the variance of a binomial r.v. is $np(1-p)$ and Y is a binomial r.v. with $n = 4$ and $p = 1/3 = P(\text{judge chooses formula } B)$.

d. $V(Y) = np(1-p) = 4\left(\frac{1}{3}\right)\left(1 - \frac{1}{3}\right) = \frac{8}{9}$.

Binomial Distribution

Example 2:

Suppose that Terry and Chris are dating. Terry's ex-boyfriend John is jealous and is snooping on Terry's text messages without Terry knowing. Because John is busy, he can only snoop each text message with a probability of 0.8, independently of the other text messages. Suppose that Terry and Chris exchanged a total of 8 text messages today. Let X be the number of those eight text messages that John read.

a. Find the probability (mass) function of X and $E(X)$.
b. Given that at least one text message was read by John, find the probability that John read exactly two text messages.
c. Suppose that if John has read least 6 of the text messages, he can figure out Terry's plans for this weekend. Find the probability that he can figure out Terry's plans for this weekend.

Solution:

Note that X is binomial with $n = 8$ and $p = 0.8$.

a. $p(x) = \binom{8}{x}(0.8)^x(1-0.8)^{8-x}$, $x = 0, 1, \ldots, 8$, and $E(X) = np = 8(0.8) = 6.4$.

pmf of binom: $p(y) = \binom{n}{y} p^y(1-p)^{n-y}$

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Solution:

Note that X is binomial with $n = 8$ and $p = 0.8$.

b. $P(X = 2 | X \geq 1) = \frac{P(X=2, X\geq 1)}{P(X\geq 1)} = \frac{P(X=2)}{P(X\geq 1)} = \frac{P(X=2)}{1-P(X=0)} = \frac{\binom{8}{2}(0.8)^2(1-0.8)^{8-2}}{1-(0.2)^8} = \frac{28(0.64)(0.000064)}{1-0.00000256} = \frac{0.00114688}{0.9999974} \approx 0.0011.$

Binomial Distribution

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Solution:

Note that X is binomial with $n = 8$ and $p = 0.8$.

c. $P(X \geq 6) = \sum_{x=6}^{8} p(x) = \sum_{x=6}^{8} \binom{8}{x} (0.8)^x (1 - 0.8)^{8-x} =$
$28(0.26)(0.04) + 8(0.21)(0.2) + 0.167 \approx 0.79.$

Bernoulli vs. Binomial Distributions

	Bernoulli	**Binomial**
Usage	single trial	multiple trials
Parameters	p (success probability)	n (num. of trials), p (success probability)
Values Y can take	$y = 0, 1$	$y = 0, 1, 2, \ldots, n$
PMF	$p(y) = p^y(1-p)^{1-y}$	$p(y) = \binom{n}{y} p^y(1-p)^{n-y}$
Mean	p	np
Variance	$p(1-p)$	$np(1-p)$
Example	a lightbulb working or not	num. of successes in 10 lightbulb tests

Geometric Distribution

Definition 3.8: Geometric Distribution

A random variable Y is said to have a *geometric distribution* if and only if

$$p(y) = (1-p)^{y-1}p,$$

where $y = 1, 2, \ldots$ and $0 \leq p \leq 1$.

- the probability distribution of a random variable counting the number of failures before first success
- We call $p(y) = (1-p)^{y-1}p$ as the pmf of a geometric random variable.
- Notation: $Y \sim \mathrm{G}(p)$, read as: "Y is a geometric random variable with probability of success p."

Geometric Distribution

Theorem 3.8

If Y is a random variable with a geometric distribution. Then

$$\mu = E(Y) = \frac{1}{p} \quad \text{and} \quad \sigma^2 = V(Y) = \frac{1-p}{p^2}.$$

Proof:

Let $q = 1 - p$.

$$\begin{aligned} E(Y) &= \sum_y y p(y) && \text{defn of expected value} \\ &= \sum_{y=1}^{\infty} y q^{y-1} p && \text{pmf of geometric distribution} \\ &= p \sum_{y=1}^{\infty} y q^{y-1} && \text{factor out the } p \text{ that doesn't depend on } y \end{aligned}$$

(continued next slide...)

Geometric Distribution

Proof:

Recall: $\frac{d}{dq}(q^y) = yq^{y-1} \Rightarrow \frac{d}{dq}\left(\sum_{y=1}^{\infty} q^y\right) = \sum_{y=1}^{\infty} yq^{y-1}$.

$$
\begin{aligned}
E(Y) &= p\sum_{y=1}^{\infty} yq^{y-1} \\
&= p\frac{d}{dq}\left(\sum_{y=1}^{\infty} q^y\right) && \text{substitute with the derived expression above} \\
&= p\frac{d}{dq}\left(\frac{q}{1-q}\right) && \text{sum of geometric series with common ratio } 0 \le q \le 1 \\
&= p\left\{\frac{1-q-q(-1)}{(1-q)^2}\right\} && \text{derivative} \\
&= \frac{p}{(1-q)^2} \\
&= \frac{p}{p^2} && q = 1-p \\
&= \frac{1}{p}.
\end{aligned}
$$

Geometric Distribution

Proof:

Recall: $\frac{d^2}{dq^2}(q^y) = y(y-1)q^{y-2} \Rightarrow \frac{d^2}{dq^2}\left(\sum_{y=2}^{\infty} q^y\right) = \sum_{y=2}^{\infty} y(y-1)q^{y-2}$.

(start summation index at $y = 2$ since $y = 1$ does not contribute to the summation)

$$
\begin{aligned}
E\{Y(Y-1)\} &= \sum_y y(y-1)p(y) && \text{defn of expected value} \\
&= \sum_{y=1}^{\infty} y(y-1)q^{y-1}p && \text{pmf of geometric distribution} \\
&= pq\sum_{y=1}^{\infty} y(y-1)q^{y-2} && \text{factor out } pq \\
&= pq\frac{d^2}{dq^2}\left(\sum_{y=2}^{\infty} q^y\right) && \text{substitute with the derived expression above} \\
&= pq\frac{d^2}{dq^2}\left(\frac{q^2}{1-q}\right) && \text{sum of geometric series with common ratio } 0 \le q \le 1 \\
&= pq\frac{d}{dq}\left\{\frac{(1-q)(2q)-q^2(-1)}{(1-q)^2}\right\} && \text{first derivative}
\end{aligned}
$$

(continued next slide...)

Geometric Distribution

Proof:

$$
\begin{aligned}
E\{Y(Y-1)\} &= pq\frac{d}{dq}\left\{\frac{(1-q)(2q)-q^2(-1)}{(1-q)^2}\right\} \\
&= pq\frac{d}{dq}\left\{\frac{2q-2q^2+q^2}{(1-q)^2}\right\} = pq\frac{d}{dq}\left\{\frac{2q-q^2}{(1-q)^2}\right\} \\
&= pq\left\{\frac{(1-q)^2(2-2q)-(2q-q^2)\{-2(1-q)\}}{(1-q)^4}\right\}
\end{aligned}
$$

second derivative

$$
\begin{aligned}
&= pq\left\{\frac{(1-q)(2-2q)+2(2q-q^2)}{(1-q)^3}\right\} \quad \text{cancel } 1-q \text{ in num \& denom} \\
&= pq\left\{\frac{2-2q-2q+2q^2+4q-2q^2}{(1-q)^3}\right\} \\
&= \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}. \quad q = 1-p
\end{aligned}
$$

(continued next slide...)

Geometric Distribution

Proof:

We have $E\{Y(Y-1)\} = \frac{2q}{p^2}$.

But we need $E(Y^2)$ which is:

$$\begin{aligned} E(Y^2) &= E\{Y(Y-1)\} + E(Y) \\ &= \frac{2q}{p^2} + \frac{1}{p}. \end{aligned}$$

We can now solve for $V(Y)$ as follows:

$$\begin{aligned} V(Y) &= E(Y^2) - \{E(Y)\}^2 \\ &= \frac{2q}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 \\ &= \frac{2q + p - 1}{p^2} \\ &= \frac{2(1-p) + p - 1}{p^2} \qquad q = 1 - p \\ &= \frac{2 - 2p + p - 1}{p^2} = \frac{1-p}{p^2}. \end{aligned}$$

Questions?

Homework Exercises: 3.37, 3.43, 3.55, 3.57, 3.65

Solutions will be discussed this Friday by the TA.