STAT 3375Q: Introduction to Mathematical Statistics I

Lecture 9: Continuous Random Variables, Expected Value

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February 21, 2024

Outline

1. Previously...
   - Cumulative Distribution Function
   - Continuous Random Variables

2. Expected Value of Continuous Random Variables

Previously...

Cumulative Distribution Function

- Cumulative Distribution Function (CDF) of Y: (discrete/continuous)

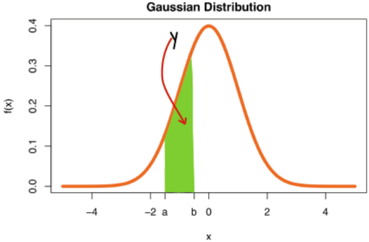
$$F(y) = P(Y \leq y), \quad -\infty < y < \infty.$$

- CDF must satisfy:
  1. $F(-\infty) = \lim_{y \to -\infty} F(y) = 0.$
  2. $F(\infty) = \lim_{y \to \infty} F(y) = 1.$
  3. $F(y)$ is a nondecreasing function of y.
     If y_1 and y_2 are any values such that $y_1 < y_2$, then $F(y_1) \leq F(y_2)$.
- Y is continuous if $F(y)$ is continuous for $-\infty < y < \infty$.

Continuous Random Variables

- Probability that Y falls in the interval $[a, b]$ is

$$P(a \leq Y \leq b) = \int_a^b f(y)dy.$$



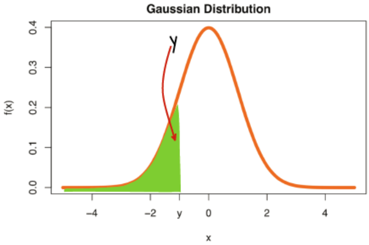
The probability that Y will fall inside the interval $[a, b]$ is simply the area under the PDF bounded by $x = 0$ (below), a (left), and b (right).

- Probability Density Function (PDF) of Y:

$$f(y) = \frac{dF(y)}{dy} = F'(y).$$

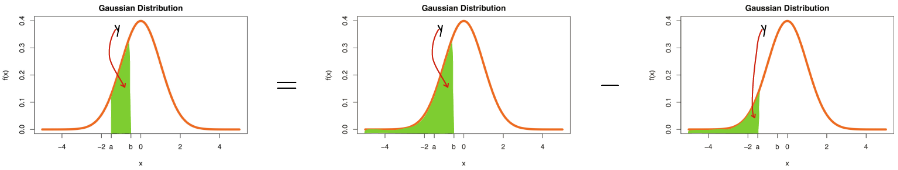
- PDF must satisfy:
  1. $f(y) \geq 0$ for all y, $-\infty < y < \infty$. $f(y)$ itself is NOT a probability!
  2. $\int_{-\infty}^{\infty} f(y)dy = 1$.
- PDF to CDF:

$$F(y) = \int_{-\infty}^{y} f(t)dt.$$



$F(y)$ is the area under the PDF bounded by $x = 0$ (below), from $-\infty$ up to y.

- Other important equalities:
  - $P(Y = y) = 0$ Why? Recall: probability $= \frac{\text{number of desired outcomes}}{\text{total number of all possible outcomes}} = \frac{1}{\infty} = 0$
  - $P(a < Y < b) = P(a \leq Y < b) = P(a < Y \leq b) = P(a \leq Y \leq b)$
  - $P(a < Y \leq b) = P(Y \leq b) - P(Y \leq a) = F(b) - F(a) = \int_a^b f(y)dy$



Continuous Random Variables

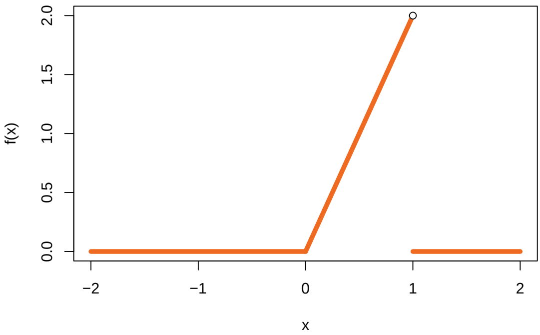
Example 1:

Let X have pdf

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Draw $f(x)$.

Solution:



Continuous Random Variables

Example 1:
Let X have pdf

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(b) Find and draw the CDF of X.

Solution:

- When $x \leq 0$,

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0dt = 0.$$

- When $x \geq 1$,

$$\begin{aligned} F(x) = \int_{-\infty}^{x} f(t)dt &= \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt \\ &= \int_{-\infty}^{0} 0dt + \int_{0}^{1} 2tdt + \int_{1}^{x} 0dt \\ &= \int_{0}^{1} 2tdt = t^2\Big|_0^1 = 1. \end{aligned}$$

(Cont'd in the next slide...)

Continuous Random Variables

Example 1:

Let X have pdf

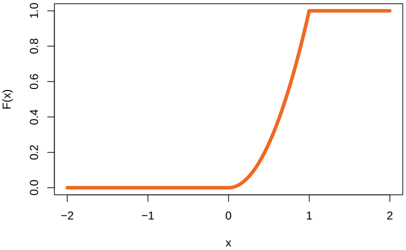
$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(b) Find and draw the CDF of X.

Solution:

- When $0 < x < 1$,

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{0}^{x} f(t)dt \\ &= \int_{0}^{x} 2tdt = t^2\Big|_0^x = x^2. \end{aligned}$$



Thus, the CDF of X is

$$F(x) = \begin{cases} 0, & \text{for } x \le 0, \\ x^2, & \text{for } 0 < x < 1, \\ 1, & \text{for } x \ge 1. \end{cases}$$

Continuous Random Variables

Example 1:

Let X have pdf

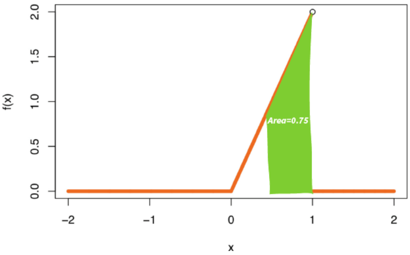
$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(c) Find $P(X > 0.5)$.

Solution:

- Approach 1. (Using the CDF)

$$\begin{aligned} P(X > 0.5) &= 1 - P(X \leq 0.5) \\ &= 1 - F(0.5) \\ &= 1 - (0.5)^2 = 0.75. \end{aligned}$$



- Approach 2. (Using the PDF)

$$\begin{aligned} P(X > 0.5) &= \int_{0.5}^{\infty} f(x)dx \\ &= \int_{0.5}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx \\ &= \int_{0.5}^{1} 2xdx + \int_{1}^{\infty} 0dx \\ &= x^2\Big|_{0.5}^{1} = 1 - 0.5^2 = 0.75. \end{aligned}$$

Continuous Random Variables

Example 1:

Let X have pdf

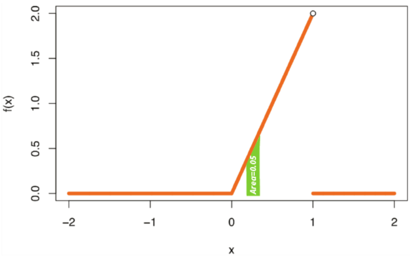
$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(d) Find $P(0.2 < X < 0.3)$.

Solution:

- Approach 1. (Using the CDF)

$$\begin{aligned} P(0.2 < X < 0.3) &= F(0.3) - F(0.2) \\ &= (0.3)^2 - (0.2)^2 \\ &= 0.05. \end{aligned}$$



- Approach 2. (Using the PDF)

$$\begin{aligned} P(0.2 < X < 0.3) &= \int_{0.2}^{0.3} f(x)dx \\ &= \int_{0.2}^{0.3} 2xdx \\ &= x^2\Big|_{0.2}^{0.3} \\ &= (0.3)^2 - (0.2)^2 \\ &= 0.05. \end{aligned}$$

Example 1:

Let X have pdf

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(e) What is the probability that X is at least $\frac{3}{4}$, given that X is at least $\frac{1}{2}$?

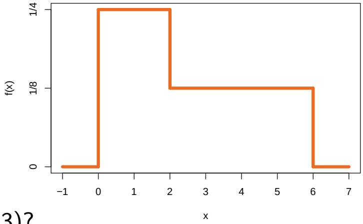
Solution:

$$\begin{aligned} P(X \geq 3/4 | X \geq 1/2) &= \frac{P\{(X \geq 3/4) \cap (X \geq 1/2)\}}{P(X \geq 1/2)} \\ &= \frac{P(X \geq 3/4)}{P(X \geq 1/2)} \\ &= \frac{\int_{3/4}^{1} 2x dx}{\int_{1/2}^{1} 2x dx} = \frac{x^2\big|_{3/4}^{1}}{x^2\big|_{1/2}^{1}} = \frac{1 - \frac{9}{16}}{1 - \frac{1}{4}} = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{12}. \end{aligned}$$

Continuous Random Variables

Example 2:

Suppose X is a continuous random variable taking values between 0 and 6 with PDF shown below.



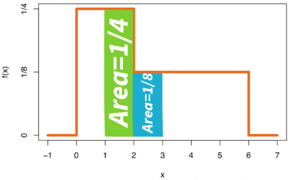
a What is $P(1 \leq X \leq 3)$?

Solution:

- Approach 1. (Using the PDF)

$$\begin{aligned} P(1 \leq X \leq 3) &= \int_1^3 f(x)dx \\ &= \int_1^2 f(x)dx + \int_2^3 f(x)dx \\ &= \int_1^2 \tfrac{1}{4}dx + \int_2^3 \tfrac{1}{8}dx \\ &= \tfrac{1}{4}x\Big|_1^2 + \tfrac{1}{8}x\Big|_2^3 \\ &= \tfrac{1}{4}(2-1) + \tfrac{1}{8}(3-2) = \tfrac{3}{8}. \end{aligned}$$

- Approach 2. (Area of rectangles)



$$P(1 \leq X \leq 3) = \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{3}{8}.$$

Continuous Random Variables

Example 3:

A continuous random variable X has probability density function:
$f(x) = ce^{-|x|/2}$.

a. Find c.
b. Find the CDF.

Continuous Random Variables

Example 3:

A continuous random variable X has probability density function: $f(x) = ce^{-|x|/2}$.

(a) Find c.

Solution:

PDF: $f(x) = ce^{-|x|/2} = \begin{cases} ce^{-x/2}, & x \geq 0, \\ ce^{x/2}, & x < 0. \end{cases}$

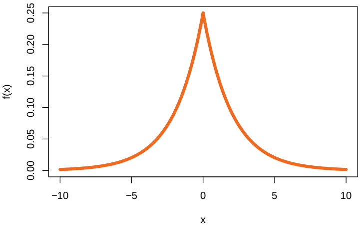
To be a valid PDF, we need $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} ce^{x/2}dx + \int_{0}^{\infty} ce^{-x/2}dx$$

$$= 2ce^{x/2}\Big|_{-\infty}^{0} - 2ce^{-x/2}\Big|_{0}^{\infty} = 4c.$$

$\Rightarrow 4c = 1$

$\Rightarrow c = \frac{1}{4}$.



Continuous Random Variables

Example 3:

A continuous random variable X has probability density function: $f(x) = ce^{-|x|/2}$.

(b) Find the CDF.

Solution:

PDF: $f(x) = \frac{1}{4}e^{-|x|/2} = \begin{cases} \frac{1}{4}e^{-x/2}, & x \geq 0, \\ \frac{1}{4}e^{x/2}, & x < 0. \end{cases}$

- When $x < 0$,

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{4}e^{t/2}dt = \frac{1}{4}(2)e^{t/2}\Big|_{-\infty}^{x} = \frac{1}{2}e^{x/2}.$$

- When $x \geq 0$,

$$\begin{aligned} F(x) = \int_{-\infty}^{x} f(t)dt &= \int_{-\infty}^{0} \frac{1}{4}e^{t/2}dt + \int_{0}^{x} \frac{1}{4}e^{-t/2}dt \\ &= \frac{1}{4}(2)e^{t/2}\Big|_{-\infty}^{0} - \frac{1}{4}(2)e^{-t/2}\Big|_{0}^{x} \\ &= \frac{1}{2} - \frac{1}{2}e^{-x/2} + \frac{1}{2} \\ &= 1 - \frac{1}{2}e^{-x/2}. \end{aligned}$$

Continuous Random Variables

Example 3:

A continuous random variable X has probability density function: $f(x) = ce^{-|x|/2}$.

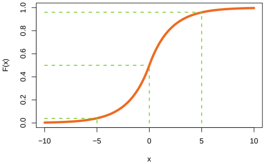
(b) Find the CDF.

Solution:

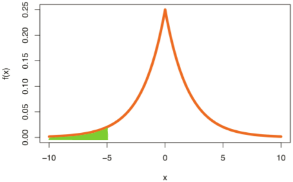
PDF: $f(x) = \frac{1}{4}e^{-|x|/2} = \begin{cases} \frac{1}{4}e^{-x/2}, & x \geq 0, \\ \frac{1}{4}e^{x/2}, & x < 0. \end{cases}$

Thus, the CDF of X is

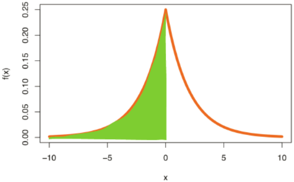
$$F(x) = \begin{cases} \frac{1}{2}e^{x/2}, & \text{for } x < 0, \\ 1 - \frac{1}{2}e^{-x/2}, & \text{for } x \geq 0. \end{cases}$$



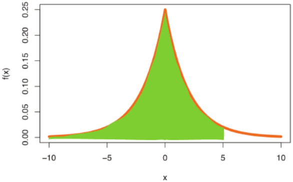
x	$F(x) = P(X \leq x)$
-5	$\frac{1}{2}e^{-5/2} = 0.04$
0	$1/2$
5	$1 - \frac{1}{2}e^{-5/2} = 0.96$



Area under PDF $= F(-5) = 0.04$



Area under PDF $= F(0) = 0.5$



Area under PDF $= F(5) = 0.96$

Expected Value of Continuous Random Variables

Expected Value of Continuous Random Variables

Definition 4.5: Expected Value

The expected value of a continuous random variable Y is

$$E(Y) = \int_{-\infty}^{\infty} yf(y)dy,$$

provided that the integral exists.

Theorem 4.4: Expected Value of Functions of Random Variables

Let $g(Y)$ be a function of Y. Then the expected value of $g(Y)$ is given by

$$E\{g(Y)\} = \int_{-\infty}^{\infty} g(y)f(y)dy,$$

provided that the integral exists.

Expected Value of Continuous Random Variables

Theorem 4.5

Let c be a constant and let $g_1(Y), g_2(Y), \ldots, g_k(Y)$ be functions of a continuous random variable Y. Then the following results hold:

1. $E(c) = c$.
2. $E\{cg(Y)\} = cE\{g(Y)\}$.
3. $E\{g_1(Y) + g_2(Y) + \ldots + g_k(Y)\} = E\{g_1(Y)\} + E\{g_2(Y)\} + \ldots + E\{g_k(Y)\}$.

Note:

- If $g(Y) = (Y - \mu)^2$, then $E\{(Y - \mu)^2\}$ is the variance of continuous random variable Y.
- More useful formula to compute the variance:

$$\sigma^2 = V(Y) = E(Y^2) - \mu^2$$

- Standard deviation:

$$\sigma = \sqrt{V(Y)}$$

Example 4:

The number of minutes between successive arrivals of airplanes at an airport is a random variable X with CDF

$$F(x) = \begin{cases} 1 - e^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

a. Find $P(5 \leq X \leq 10)$.

Solution:

$$P(5 \leq X \leq 10) = F(10) - F(5) = (1 - e^{-10/5}) - (1 - e^{-5/5}) = e^{-1} - e^{-2} = 0.2325.$$

Example 4:

The number of minutes between successive arrivals of airplanes at an airport is a random variable X with CDF

$$F(x) = \begin{cases} 1 - e^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find the probability that there are more than 10 minutes between between successive flights.

Solution:

$$P(X > 10) = 1 - P(X \leq 10) = 1 - F(10) = 1 - (1 - e^{-10/5}) = e^{-2} = 0.1353.$$

Expected Value of Continuous Random Variables

Example 4:

The number of minutes between successive arrivals of airplanes at an airport is a random variable X with CDF

$$F(x) = \begin{cases} 1 - e^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

- (c) Find the PDF for X.

Solution:

$$f(x) = F'(x) = \begin{cases} \frac{1}{5} e^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Expected Value of Continuous Random Variables

Example 4:

The number of minutes between successive arrivals of airplanes at an airport is a random variable X with cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(d) Find $E(X)$ and $V(X)$.

Solution:

Recall: $f(x) = F'(x) = \begin{cases} \frac{1}{5}e^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$

$$\begin{aligned} E(X) &= \int_0^\infty \tfrac{x}{5} e^{-x/5} dx \\ &= \left(\tfrac{x}{5}\right)\left(-5e^{-x/5}\right)\Big|_0^\infty - \int_0^\infty \left(-5e^{-x/5}\right)\left(\tfrac{1}{5}dx\right) \\ &= -xe^{-x/5}\Big|_0^\infty + \int_0^\infty e^{-x/5} dx \\ &= \int_0^\infty e^{-x/5} dx = -5e^{-x/5}\Big|_0^\infty = 5. \end{aligned}$$

Integration by parts:

$\int_a^b u dv = uv|_a^b - \int_a^b v du$

$u = \frac{x}{5}$,

$du = \frac{1}{5}dx$,

$dv = e^{-x/5}dx$,

$v = \int e^{-x/5}dx = -5e^{-x/5}$.

****

Note: Since $-xe^{-x/5}\Big|_0^\infty = \lim_{x\to\infty} -\frac{x}{e^{x/5}} = \frac{\infty}{\infty}$, apply L'Hospital's rule.

$\Rightarrow -xe^{-x/5}\Big|_0^\infty = \lim_{x\to\infty} -\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^{x/5})} = \lim_{x\to\infty} -\frac{1}{\frac{1}{5}e^{x/5}} = 0.$

Expected Value of Continuous Random Variables

Example 4:

The number of minutes between successive arrivals of airplanes at an airport is a random variable X with cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$$\begin{aligned} E(X^2) &= \int_0^\infty \frac{x^2}{5} e^{-x/5} dx \\ &= \left(\frac{x^2}{5}\right)\left(-5e^{-x/5}\right)\Big|_0^\infty - \int_0^\infty \left(-5e^{-x/5}\right)\left(\frac{2x}{5}dx\right) \\ &= -x^2 e^{-x/5}\Big|_0^\infty + \int_0^\infty 2xe^{-x/5}dx \end{aligned}$$

Integration by parts:
$\int_a^b u dv = uv|_a^b - \int_a^b v du$
$u = \frac{x^2}{5}$,
$du = \frac{2x}{5} dx$,
$dv = e^{-x/5} dx$,
$v = \int e^{-x/5} dx = -5e^{-x/5}$.

$$\begin{aligned} &= 2\int_0^\infty xe^{-x/5}dx \\ &= 2\left\{x\left(-5e^{-x/5}\right)\Big|_0^\infty - \int_0^\infty \left(-5e^{-x/5}\right)dx\right\} \\ &= (2)(5)\left(\int_0^\infty e^{-x/5}dx\right) = (2)(5)\left(-5e^{-x/5}\Big|_0^\infty\right) \\ &= 50. \end{aligned}$$

Integration by parts:
$\int_a^b u dv = uv|_a^b - \int_a^b v du$
$u = x$,
$du = dx$,
$dv = e^{-x/5} dx$,
$v = \int e^{-x/5} dx = -5e^{-x/5}$.

****

Note: Since $-x^2 e^{-x/5}\Big|_0^\infty = \lim_{x\to\infty} -\frac{x^2}{e^{x/5}} = \frac{\infty}{\infty}$, apply L'Hospital's rule. This will be zero.

Example 4:

The number of minutes between successive arrivals of airplanes at an airport is a random variable X with cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$$V(X) = E(X^2) - \{E(X)\}^2 = 50 - 5^2 = 25.$$

Example 5:

Alex came up with a function that he thinks could represent a probability density function. He defined the potential PDF for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid PDF? If not, find a constant c such that the PDF $f(x) = \frac{c}{1+x^2}$ is valid. Then find $E(X)$.

Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad \tan \frac{\pi}{2} = \infty, \quad \tan 0 = 0$

Expected Value of Continuous Random Variables

Example 5:

Alex came up with a function that he thinks could represent a probability density function. He defined the potential PDF for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid PDF? If not, find a constant c such that the PDF $f(x) = \frac{c}{1+x^2}$ is valid. Then find $E(X)$.

Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, $\tan 0 = 0$

Solution:

- Is this a valid PDF?

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \frac{1}{1+x^2} dx = \tan^{-1} x \Big|_0^\infty = \tan^{-1}(\infty) - \tan^{-1}(0) = \frac{\pi}{2} \neq 1.$$

Not a valid PDF.

Expected Value of Continuous Random Variables

Example 5:

Alex came up with a function that he thinks could represent a probability density function. He defined the potential PDF for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid PDF? If not, find a constant c such that the PDF $f(x) = \frac{c}{1+x^2}$ is valid. Then find $E(X)$.

Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, $\tan 0 = 0$

Solution:

- Solving for c that will make this function a valid PDF:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \frac{c}{1+x^2}dx = c\tan^{-1} x\Big|_0^{\infty} = c\{\tan^{-1}(\infty) - \tan^{-1}(0)\} = \frac{c\pi}{2}.$$

We need $\frac{c\pi}{2} = 1$. Thus, $c = \frac{2}{\pi}$.

Expected Value of Continuous Random Variables

Example 5:

Alex came up with a function that he thinks could represent a probability density function. He defined the potential PDF for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid PDF? If not, find a constant c such that the PDF $f(x) = \frac{c}{1+x^2}$ is valid. Then find $E(X)$.

Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan\frac{\pi}{2} = \infty$, $\tan 0 = 0$

Solution:

- Computing the expected value:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} \frac{2x}{\pi(1+x^2)}dx = \frac{2}{\pi}\int_0^{\infty} \frac{x}{(1+x^2)}dx$$
$$= \frac{1}{\pi}\ln(1+x^2)\Big|_0^{\infty} = \infty.$$

The expected value or mean is undefined!
This is the standard Cauchy distribution, known for having heavier tails than a Gaussian distribution.

Expected Value of Continuous Random Variables

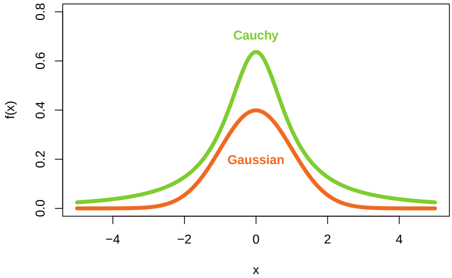
Example 5:

Alex came up with a function that he thinks could represent a probability density function. He defined the potential PDF for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid PDF? If not, find a constant c such that the PDF $f(x) = \frac{c}{1+x^2}$ is valid. Then find $E(X)$.

Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, $\tan 0 = 0$

Solution:

Standard Cauchy distribution: $f(x) = \frac{2x}{\pi(1+x^2)}$



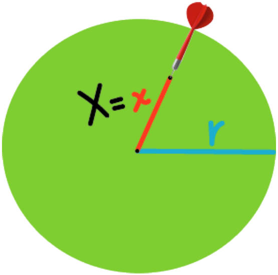
Expected Value of Continuous Random Variables

Example 6:

Consider the closed unit circle of radius r, i.e., $S = \{(x, y) : x^2 + y^2 \leq r^2\}$. Suppose we throw a dart onto this circle and are guaranteed to hit it, but the dart is equally likely to land anywhere in S. This means that the probability that the dart lands in any particular area of size A (that is entirely inside the circle of radius r) is equal to $\frac{A}{\text{Area of whole circle}}$. The density outside the circle of radius r is 0.

Let X be the distance the dart lands from the center. What is the CDF and PDF of X? What is $E(X)$ and $V(X)$?

Solution:



- $F(x) = P(X \leq x)$ is simply the probability that the dart lands inside the circle of radius x.
- Given: probability dart lands in an area of size A $= \frac{A}{\text{Area of whole circle}} = \frac{\pi x^2}{\pi r^2}$.

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{r^2}, & 0 < x \leq r \\ 1, & x > r. \end{cases}$$

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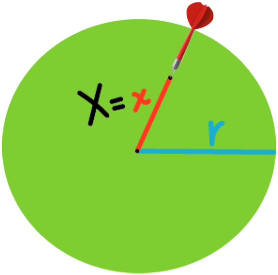

- Solving for the PDF:

$$f(x) = F'(x) = \begin{cases} \frac{2x}{r^2}, & 0 < x \leq r \\ 0, & \text{otherwise.} \end{cases}$$

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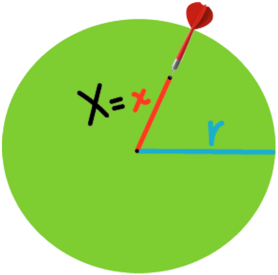

- $E(X) = \int_{\infty}^{-\infty} xf(x)dx = \int_0^r x\frac{2x}{r^2}dx = \frac{2}{3r^2}x^3\Big|_0^r = \frac{2}{3}r.$
- $E(X^2) = \int_{\infty}^{-\infty} x^2 f(x)dx = \int_0^r x^2\frac{2x}{r^2}dx = \frac{2}{4r^2}x^4\Big|_0^r = \frac{1}{2}r^2.$
- $V(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{2}r^2 - \left(\frac{2}{3}r\right)^2 = \frac{1}{18}r^2.$

Questions?

Homework Exercises: 4.5, 4.9, 4.13, 4.15, 4.17

Solutions will be discussed this Friday by the TA.