Department of Statistics
College of Liberal Arts & Sciences

STAT 3375Q: Introduction to Mathematical Statistics I

Spring 2024

Midterm 1 Simulation Solutions Date: 12 February 2024

Problem 1
Suppose $P(A) > 0$ and $P(B) > 0$. Prove that if $P(A|B) > P(A)$ then $P(B|A) > P(B)$. *(20 points)*

Solution:

$$
\begin{aligned}
P(A|B) &> P(A) && \text{(given)} \\
\Rightarrow \frac{P(A \cap B)}{P(B)} &> P(A) && \text{(defn of conditional prob)} \\
\Rightarrow \frac{P(B|A)P(A)}{P(B)} &> P(A) && P(B|A) = \frac{P(A \cap B)}{P(A)} \\
\Rightarrow \frac{P(B|A)P(A)}{P(A)} &> P(B) && \text{(sign of inequality does not flip since } P(A) \text{ and } P(B) \text{ are positive)} \\
\Rightarrow P(B|A) &> P(B) && \text{cancel } P(A) \text{ in numerator and denominator}
\end{aligned}
$$

Problem 2
Let $P(A) = 0.3$ and $P(B) = 0.6$.

a) Find $P(A \cup B)$ when A and B are independent. *(6 points)*

b) Find $P(A|\bar{B})$ when A and B are independent. *(7 points)*

c) Find $P(A|B)$ when A and B are mutually exclusive. *(7 points)*

Solution:

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (additive law of probability)
$= 0.3 + 0.6 - P(A)P(B)$
$= 0.9 - (0.3)(0.6)$
$= 0.9 - 0.18$
$= 0.72$

b) $P(A|\bar{B}) = \frac{P(A\cap\bar{B})}{P(\bar{B})} = \frac{P(A)-P(A\cap B)}{1-P(B)}$ (total probability: $P(A) = P(A \cap B) + P(A \cap \bar{B})$)
$= \frac{0.3-P(A)P(B)}{1-0.6}$
$= \frac{0.3-(0.3)(0.6)}{0.4}$
$= \frac{0.12}{0.4} = 0.3.$

c) $P(A|B) = \frac{P(A\cap B)}{P(B)} = 0.$

Problem 3

Five cards are dealt at random and without replacement from a standard deck of 52 cards. Note that there are 13 heart cards in the deck.

a) What is the probability that the hand contains 5 hearts? *(6 points)*

b) What is the probability that the hand contains at least 4 hearts? *(7 points)*

c) What is the probability that the hand contains 5 hearts if it is known that the hand contains at least 4 hearts? *(7 points)*

Solution:

the number of ways to select 5 cards: $\binom{52}{5}$

the number of ways to select 5 hearts: $\binom{13}{5}$

the number of ways to select 4 hearts and one other card: $\binom{13}{4}\binom{39}{1}$

Suppose H is the event of picking a heart card and O is the event of picking a card that is not a heart.

a) $P(H_1H_2H_3H_4H_5) = \dfrac{\binom{13}{5}}{\binom{52}{5}} = \frac{1287}{2598960} = 0.000495.$

b) $P(H_1H_2H_3H_4O_5 \cup H_1H_2H_3H_4H_5) = \dfrac{\binom{13}{4}\binom{39}{1} + \binom{13}{5}}{\binom{52}{5}} = \frac{(715)(39)+1287}{2598960} = 0.0112.$

c) $P(H_1H_2H_3H_4H_5|H_1H_2H_3H_4O_5 \cup H_1H_2H_3H_4H_5)$
$= \frac{P\{H_1H_2H_3H_4H_5 \cap (H_1H_2H_3H_4O_5 \cup H_1H_2H_3H_4H_5)\}}{P(H_1H_2H_3H_4O_5 \cup H_1H_2H_3H_4H_5)}$
$= \frac{P\{H_1H_2H_3H_4H_5\}}{0.0112} = \frac{0.000495}{0.0112} = 0.044.$

Problem 4

Assume that an insurance company knows the following probabilities relating to automobile accidents (where the second column refers to the probability that the policyholder has at least one accident during the annual policy period):

Age of Driver	Probability of Accident	Fraction of Company's Insured Drivers
16-25	0.05	0.10
26-50	0.02	0.55
51-65	0.03	0.20
66-90	0.04	0.15

A randomly selected driver from the company's insured drivers has an accident. What is the probability that the driver is in the 16–25 age group? *(20 points)*

Solution:

Define the following events:

$$\begin{aligned} R_1 &= \text{“ages 16-25,”} \\ R_2 &= \text{“ages 26-50,”} \\ R_3 &= \text{“ages 51-65,”} \\ R_4 &= \text{“ages 66-90.”} \end{aligned}$$

We are given the probabilities of selecting a driver from each category:

$$P(R_1) = 0.10, \quad P(R_2) = 0.55, \quad P(R_3) = 0.2 \quad \text{and} \quad P(R_4) = 0.15.$$

Let A be the event that a chosen driver gets in an accident in a given year.
From the table, we know the probabilities:

$$P(A|R_1) = 0.05, \quad P(A|R_2) = 0.02, \quad P(A|R_3) = 0.03 \quad \text{and} \quad P(A|R_4) = 0.04.$$

Using Bayes' rule, we can compute for $P(R_1|A)$ as follows:

$$\begin{aligned} P(R_1|A) &= \frac{P(A|R_1)P(R_1)}{P(A|R_1)P(R_1) + P(A|R_2)P(R_2) + P(A|R_3)P(R_3) + P(A|R_4)P(R_4)} \\ &= \frac{(0.05)(0.10)}{(0.05)(0.10) + (0.02)(0.55) + (0.03)(0.2) + (0.04)(0.15)} \\ &= 0.179. \end{aligned}$$

Problem 5
A balanced die is thrown 5 times. Let X be the number of times that a number smaller than 3 had shown up.

a) Find the distribution function of X. *(10 points)*

b) Find the distribution function of $Y = 2X - 1$. *(10 points)*

Solution:

a) X is a binomial random variable with $n = 5$ and $p = \frac{2}{6} = \frac{1}{3}$.
The probability mass function of X is:

$$P(X = x) = p(x) = \binom{5}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{5-x}, \quad 0 \leq x \leq 5.$$

b)

$$\begin{aligned} P(Y = y) &= P\left(2X - 1 = y\right) \\ &= P\left(2X = 1 + y\right) \\ &= P\left(X = \frac{1+y}{2}\right) \\ &= p\left(\frac{1+y}{2}\right) \\ &= \binom{5}{\frac{1+y}{2}}\left(\frac{1}{3}\right)^{\frac{1+y}{2}}\left(\frac{2}{3}\right)^{5-\left(\frac{1+y}{2}\right)}, \quad -1 \leq y \leq 9. \end{aligned}$$

Note: The x values are $0, 1, 2, 3, 4, 5$ while the y values are $-1, 1, 3, 5, 7, 9$.

x	$p(x)$
0	32/243
1	80/243
2	80/243
3	40/243
4	10/243
5	1/243

y	$p(y)$
-1	32/243
1	80/243
3	80/243
5	40/243
7	10/243
9	1/243

Problem 6

If the random variable X has a geometric distribution such that $V(X)/E(X) = a$, where $a > 0$.

a) What $E(X)$? Express your answer in terms of a. *(6 points)*

b) If $a = 1$, what is $P(X > 4)$? *(7 points)*

c) If $a = 1$, what is $P(X > 7|X > 3)$? *(7 points)*

Solution:

a)
- Since X has a geometric distribution, $E(X) = \frac{1}{p}$ and $V(X) = \frac{1-p}{p^2}$. (p is the probability of success.)
- $\Rightarrow \frac{V(X)}{E(X)} = \frac{\frac{1-p}{p^2}}{\frac{1}{p}} = \frac{1-p}{p} = a$ (given)
- Solving for p, we get
  $\Rightarrow 1 - p = pa$
  $\Rightarrow 1 = pa + p$
  $\Rightarrow 1 = p(a + 1)$
  $\Rightarrow p = \frac{1}{a+1}$.
- Solving for $E(X)$, we get
  $\Rightarrow E(X) = \frac{1}{p} = \frac{1}{\frac{1}{a+1}} = a + 1$.

b)
- If $a = 1$, then $p = \frac{1}{2}$.
- $P(X > 4) = p(5) + p(6) + p(7) + \dots$
  $= \left(1 - \frac{1}{2}\right)^4 \left(\frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)^5 \left(\frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)^6 \left(\frac{1}{2}\right) + \dots$
  (pmf of geometric distribution: $p(x) = (1-p)^{x-1}p$)
  $= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^7 + \dots$
  (sum of geometric sequence with first term $a_1 = \left(\frac{1}{2}\right)^5$ and common ratio $r = \frac{1}{2}$: $S = \frac{a_1}{1-r}$)
  $= \frac{\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^5}{\frac{1}{2}} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

c)
- If $a = 1$, then $p = \frac{1}{2}$.
- $P(X > 7|X > 3) = \frac{P(X>7 \text{ and } X>3)}{P(X>3)} = \frac{P(X>7)}{P(X>3)}$
  $= \frac{p(8)+p(9)+p(10)+\dots}{p(4)+p(5)+p(6)+\dots}$
  $= \frac{\left(1-\frac{1}{2}\right)^7\left(\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)^8\left(\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)^9\left(\frac{1}{2}\right)+\dots}{\left(1-\frac{1}{2}\right)^3\left(\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)^4\left(\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)^5\left(\frac{1}{2}\right)+\dots}$
  $= \frac{\left(\frac{1}{2}\right)^8+\left(\frac{1}{2}\right)^9+\left(\frac{1}{2}\right)^{10}+\dots}{\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^6+\dots}$
  $= \frac{\frac{\left(\frac{1}{2}\right)^8}{1-\frac{1}{2}}}{\frac{\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}}} = \frac{\frac{\left(\frac{1}{2}\right)^8}{\frac{1}{2}}}{\frac{\left(\frac{1}{2}\right)^4}{\frac{1}{2}}} = \frac{\left(\frac{1}{2}\right)^7}{\left(\frac{1}{2}\right)^3} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

Problem 7

Four busses carrying 148 students arrive at a school. The busses carry 40, 33, 25 and 50 students. A student is selected at random from among the 148. Let X be the number of students on the bus of the randomly selected student. A random driver (from among the four drivers) is selected at random. Let Y be the number of students on the bus of the randomly selected driver.

a) Compute $E(X)$. *(10 points)*

b) Compute $E(Y)$. *(10 points)*

Solution:

a)

x	$p(x)$
40	40/148
33	33/148
25	25/148
50	50/148

$E(X) = 40\left(\frac{40}{148}\right) + 33\left(\frac{33}{148}\right) + 25\left(\frac{25}{148}\right) + 50\left(\frac{50}{148}\right) \approx 39.28.$

b)

y	$p(y)$
40	1/4
33	1/4
25	1/4
50	1/4

$E(Y) = 40\left(\frac{1}{4}\right) + 33\left(\frac{1}{4}\right) + 25\left(\frac{1}{4}\right) + 50\left(\frac{1}{4}\right) = 37.$