Department of Statistics
College of Liberal Arts & Sciences

STAT 3375Q: Introduction to Mathematical Statistics I

Spring 2024

Midterm 1 Exam Date: 14 February 2024

Problem 1
A pair of events A and B cannot be simultaneously mutually exclusive and independent. Prove that if $P(A) > 0$ and $P(B) > 0$, then:

a) If A and B are mutually exclusive, they cannot be independent. *(10 points)*

b) If A and B are independent, they cannot be mutually exclusive. *(10 points)*

Solution:

a) Given: A and B are mutually exclusive.
$\Rightarrow A \cap B = \emptyset$.
$\Rightarrow P(A \cap B) = 0$.
If A and B are independent (p argument), then $P(A) = 0$ or $P(B) = 0$ (q argument) since $P(A \cap B) = 0$ and $P(A \cap B) = P(A)P(B)$.
Since it is given that $P(A) > 0$ and $P(B) > 0$ (-q), then A and B cannot be independent (-p).

b) Given: A and B are independent.
$\Rightarrow P(A \cap B) = P(A)P(B)$.
$\Rightarrow P(A \cap B) > 0$ since $P(A) > 0$ and $P(B) > 0$.
$\Rightarrow P(A \cap B) \neq 0$.
$\Rightarrow A$ and B are not mutually exclusive.

Problem 2
Prove each of the following statements. (Assume that any conditioning event has positive probability.)

a) If $P(B) = 1$, then $P(A|B) = P(A)$ for any A. *(5 points)*

b) If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = \frac{P(A)}{P(B)}$. *(5 points)*

c) If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}. \qquad \textit{(5 points)}$$

d) $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$. *(5 points)*

Solution:

a) $P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A \cap B)$.
Let B and $\bar{B}$ be a partition of the sample space S, i.e., $S = B \cup \bar{B}$ and $B \cap \bar{B} = \emptyset$.
By law of total probability, $P(A) = P(A \cap B) + P(A \cap \bar{B})$.
Since $(A \cap \bar{B}) \subset \bar{B}$ and $P(\bar{B}) = 1 - P(B) = 1 - 1 = 0$, $P(A \cap \bar{B}) = 0$.
Hence, $P(A) = P(A \cap B)$.
Therefore, $P(A|B) = P(A)$.

b) If $A \subset B$, then $A \cap B = A$.
$\Rightarrow P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$. Also, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$.

c) If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

$$P(A|A \cup B) = \frac{P\{A \cap (A \cup B)\}}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B)}.$$

d) $P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

Problem 3
Cards are dealt, one at a time, from a standard 52-card deck.

a) If the first 2 cards are both spades, what is the probability that the next 3 cards are also spades? *(6 points)*

b) If the first 3 cards are all spades, what is the probability that the next 2 cards are also spades? *(7 points)*

c) If the first 4 cards are all spades, what is the probability that the next card is also a spade? *(7 points)*

Solution:

a) Approach 1.
$$P(S_3S_4S_5|S_1S_2) = \frac{P(S_3 \cap S_4 \cap S_5 \cap S_1 \cap S_2)}{P(S_1 \cap S_2)}$$
$$= \frac{(\frac{13}{52})(\frac{12}{51})(\frac{11}{50})(\frac{10}{49})(\frac{9}{48})}{(\frac{13}{52})(\frac{12}{51})} = 0.084$$

Approach 2. 11 spades left, 50 cards left.
$$P(S_3S_4S_5|S_1S_2) = \frac{\binom{11}{3}}{\binom{50}{3}} = 0.084.$$

b) Approach 1.
$$P(S_4S_5|S_1S_2S_3) = \frac{P(S_3 \cap S_4 \cap S_5 \cap S_1 \cap S_2)}{P(S_1 \cap S_2 \cap S_3)}$$
$$= \frac{(\frac{13}{52})(\frac{12}{51})(\frac{11}{50})(\frac{10}{49})(\frac{9}{48})}{(\frac{13}{52})(\frac{12}{51})(\frac{11}{50})} = 0.383$$

Approach 2. 10 spades left, 49 cards left.
$$P(S_4S_5|S_1S_2S_3) = \frac{\binom{10}{2}}{\binom{49}{2}} = 0.383.$$

c) Approach 1.
$$P(S_5|S_1S_2S_3S_4) = \frac{P(S_3 \cap S_4 \cap S_5 \cap S_1 \cap S_2)}{P(S_1 \cap S_2 \cap S_3 \cap S_4)}$$
$$= \frac{(\frac{13}{52})(\frac{12}{51})(\frac{11}{50})(\frac{10}{49})(\frac{9}{48})}{(\frac{13}{52})(\frac{12}{51})(\frac{11}{50})(\frac{10}{49})} = 0.1875$$

Approach 2. 9 spades left, 48 cards left.
$$P(S_5|S_1S_2S_3S_4) = \frac{\binom{9}{1}}{\binom{48}{1}} = 0.1875.$$

Problem 4
Suppose that 5% of men and 0.25% of women are color blind. A person is chosen at random and that person is color blind. What is the probability that the person is male? (Assume males and females to be in equal numbers.) *(20 points)*

Solution:

Using Bayes' rule, we have

$$\begin{aligned} P(M|CB) &= \frac{P(CB|M)P(M)}{P(CB|M)P(M) + P(CB|F)P(F)} \\ &= \frac{(0.05)(0.5)}{(0.05)(0.5) + (0.0025)(0.5)} \\ &= 0.9524. \end{aligned}$$

Problem 5

Suppose X has the geometric PMF $p(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \ldots$. Determine the probability distribution of $Y = \frac{X}{X+1}$. Note that both X and Y are discrete random variables. To specify the probability distribution means to specify its PMF. *(20 points)*

Solution:

$$\begin{aligned}
P(Y = y) &= P\left(\frac{X}{X+1} = y\right) \\
&= P\left(X = Xy + y\right) \\
&= P\left(X - Xy = y\right) \\
&= P\left(X(1-y) = y\right) \\
&= P\left(X = \frac{y}{1-y}\right) \\
&= p(\frac{y}{1-y}) \\
&= \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}},
\end{aligned}$$

where $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{x}{x+1}, \ldots$.

Problem 6
Suppose the random variable X has a Poisson distribution such that $P(X = 1) = P(X = 2)$.

a) Find $P(X = 4)$. *(10 points)*

b) Find $P(X \geq 4|X \geq 2)$. *(10 points)*

Solution: Recall the Poisson PMF for a random variable Y: $p(y) = \frac{\lambda^y}{y!}e^{-\lambda}$.

a)

$$\begin{aligned} P(X = 1) &= P(X = 2) \\ p(1) &= p(2) \\ \frac{\lambda e^{-\lambda}}{1!} &= \frac{\lambda^2 e^{-\lambda}}{2!} \\ \lambda &= \frac{\lambda^2}{2} \\ \lambda &= 2. \end{aligned}$$

Therefore, $P(X = 4) = p(4) = \frac{2^4 e^{-2}}{4!} = \frac{2}{3}e^{-2}$.

b)

$$\begin{aligned} P(X \geq 4|X \geq 2) &= \frac{P(X \geq 4 \text{ and } X \geq 2)}{P(X \geq 2)} \\ &= \frac{P(X \geq 4)}{P(X \geq 2)} \\ &= \frac{1 - \{p(0) + p(1) + p(2) + p(3)\}}{1 - \{p(0) + p(1)\}} \\ &= \frac{1 - \{\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} + \frac{2^3 e^{-2}}{3!}\}}{1 - \{\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!}\}} \\ &= \frac{1 - e^{-2}\{\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\}}{1 - e^{-2}\{\frac{2^0}{0!} + \frac{2^1}{1!}\}} \\ &= \frac{1 - e^{-2}(1 + 2 + 2 + \frac{4}{3})}{1 - e^{-2}(1 + 2)} \\ &= \frac{1 - e^{-2}(\frac{19}{3})}{1 - e^{-2}(3)} \approx 0.24. \end{aligned}$$

Problem 7
Let Y be the number of successes throughout n independent repetitions of a random experiment having the probability of success $p = \frac{1}{4}$. Determine the smallest value of n so that $P(Y \geq 1) \geq 0.70$. *(20 points)*

Solution: Recall the binomial PMF for a random variable Y: $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$, $y = 0, 1, \ldots, n$.
Note that it is easier to solve the probabilities of the complement.

$$\begin{aligned} P(Y \geq 1) &= 1 - P(Y = 0) \\ &= 1 - \binom{n}{0} \left(\frac{1}{4}\right)^0 \left(1 - \frac{1}{4}\right)^{n-0} \\ &= 1 - \left(\frac{3}{4}\right)^n. \end{aligned}$$

We need to find n such that $1 - \left(\frac{3}{4}\right)^n \geq 0.7 \Rightarrow 0.3 \geq \left(\frac{3}{4}\right)^n \Rightarrow \log(0.3) \geq n \log\left(\frac{3}{4}\right) \Rightarrow n > \frac{\log(0.3)}{\log\left(\frac{3}{4}\right)} = 4.19$.
Therefore, n must be 5 so that $P(Y \geq 1) \geq 0.70$.

Note: We need to switch the inequality sign since $\log\left(\frac{3}{4}\right)$ is negative.