Department of Statistics
College of Liberal Arts & Sciences

# STAT 3375Q: Introduction to Mathematical Statistics I

Spring 2024

**Midterm 2** Exam Date: 3 April 2024

**Problem 1**

Suppose $X$ and $Y$ are independent Gaussian random variables. That is, $X \sim \mathcal{N}(1, 4)$ and $Y \sim \mathcal{N}(0, 7)$.

a) Find $\text{Cov}(X, Y)$. *(4 points)*

b) Find $E(X^2Y^2)$. *(4 points)*

c) Find $E(3X - 2Y)$. *(4 points)*

d) Find $V(3X - 2Y)$. *(4 points)*

e) Find $P(-3 \leq 3X - 2Y \leq 5)$. *(4 points)*
Hint: Sum of 2 Gaussian RVs is a Gaussian RV.

*Solution:*

a) Since $X$ and $Y$ are independent, $\text{Cov}(X, Y) = 0$.

b) Since $X$ and $Y$ are independent, we can split the expected value as follows:

$$E(X^2Y^2) = E(X^2)E(Y^2).$$

Solving for $E(X^2)$, we have $E(X^2) = V(X) + \{E(X)\}^2 = 4 + 1^2 = 5$.
Solving for $E(Y^2)$, we have $E(Y^2) = V(Y) + \{E(Y)\}^2 = 7 + 0 = 7$.
Therefore, $E(X^2Y^2) = E(X^2)E(Y^2) = (5)(7) = 35$.

c) By the linearity of expectation, we have

$$\begin{aligned} E(3X - 2Y) &= 3E(X) - 2E(Y) \\ &= 3(1) - 2(0) \quad X \sim \mathcal{N}(1,4) \text{ and } Y \sim \mathcal{N}(0,7) \\ &= 3. \end{aligned}$$

d)

$$\begin{aligned} V(3X - 2Y) &= V(3X) + V(-2Y) \\ &\quad \text{Variance of the sum of independent RVs: } V(X+Y) = V(X) + V(Y) \\ &= 3^2V(X) + (-2)^2V(Y) \quad \text{Variance of a linear transform: } V(aX+b) = a^2V(X) \\ &= 9(4) + 4(7) \quad X \sim \mathcal{N}(1,4) \text{ and } Y \sim \mathcal{N}(0,7) \\ &= 64. \end{aligned}$$

e)
- Let $W = 3X - 2Y$.
- From part c) and d), we know that $W \sim \mathcal{N}(\mu = 3, \sigma^2 = 64)$.

- Thus,

$$\begin{aligned}
P(-3 \leq W \leq 5) &= P\left(\frac{-3-\mu}{\sigma} \leq \frac{W-\mu}{\sigma} \leq \frac{5-\mu}{\sigma}\right) \quad \text{standardization} \\
&= P\left(\frac{-3-3}{\sqrt{64}} \leq \frac{W-3}{\sqrt{64}} \leq \frac{5-3}{\sqrt{64}}\right) \\
&= P\left(-\frac{6}{8} \leq Z \leq \frac{2}{8}\right) \\
&= \Phi\left(\frac{1}{4}\right) - \Phi\left(-\frac{3}{4}\right) \quad \text{probability = area under the standard normal curve} \\
&= 0.59871 - 0.22663 \quad \text{Z-table values} \\
&= 0.3721.
\end{aligned}$$

□

**Problem 2**
Consider a random variable $X$ with the PDF

$$f(x) = A + Bx^2, \quad 0 \le x \le 2.$$

If $E(X) = 1/2$, find $A$ and $B$. *(20 points)*

*Solution:*

- There are two unknowns, $A$ and $B$. We will need two linear equations to find their values.
- Since $f(x)$ is a valid density, it must integrate to 1.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= \int_0^2 A + Bx^2 dx \\ &= Ax + B\frac{x^3}{3}\Big|_0^2 \\ &= 2A + \frac{8}{3}B. \end{aligned}$$

- Also, we need $E(X) = 1/2$. This means

$$\begin{aligned} \frac{1}{2} &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^2 x(A + Bx^2)dx \\ &= A\frac{x^2}{2} + B\frac{x^4}{4}\Big|_0^2 \\ &= 2A + 4B. \end{aligned}$$

- Solving the following system of linear equations, we have

$$\begin{cases} 2A + \frac{8}{3}B = 1 \\ 2A + 4B = \frac{1}{2} \end{cases}$$

$$\Rightarrow \frac{4}{3}B = -\frac{1}{2} \quad \text{subtracting the 1st eqn from the 2nd}$$

$$\Rightarrow B = -\frac{3}{8}$$

$$\Rightarrow 2A + \frac{8}{3}\left(-\frac{3}{8}\right) = 1 \quad \text{substituting the value of } B \text{ to the 1st eqn}$$

$$\Rightarrow A = 1.$$

Thus, we have the following PDF:

$$f(x) = 1 - \frac{3}{8}x^2, \quad 0 \le x \le 2.$$

□

**Problem 3**

Suppose that the completion time in hours $T$ for the STAT 3375Q final exam follows a distribution with density

$$f(t) = \frac{2}{27}(t^2 + t), \quad 0 \le t \le 3.$$

What is the probability that a randomly chosen student finishes the exam during the first 30 minutes. *(20 points)*

*Solution:*

$$\begin{aligned}
P\left(T \le \frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \frac{2}{27}(t^2 + t)dt \\
&\quad \text{probability = area under density curve} \\
&= \frac{2}{27}\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Bigg|_0^{\frac{1}{2}} \\
&= \frac{2}{27}\left\{\frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{2}\right)^2\right\} \\
&= \frac{2}{27}\left(\frac{1}{24} + \frac{1}{8}\right) \\
&= \frac{2}{27}\left(\frac{4}{24}\right) \\
&= \frac{1}{81}. \quad \square
\end{aligned}$$

**Problem 4**
Given that $X$ has MGF

$$m(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t},$$

what is the probability that $X$ is even. *(20 points)*

*Solution:*
Matching the MGF above to the MGF formula $m(t) = E(e^{tX}) = \sum_y e^{tx}p(x)$, we know that the given MGF corresponds to a discrete random variable with PMF:

$$p(x) = \begin{cases} \frac{1}{6}, & \text{if } x = -2, \\ \frac{1}{3}, & \text{if } x = -1, \\ \frac{1}{4}, & \text{if } x = 1, \\ \frac{1}{4}, & \text{if } x = 2. \end{cases}$$

Therefore,

$$P(X \text{ is even}) = P(X = -2) + P(X = 2) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

□

**Problem 5**
Suppose $X$ and $Y$ are continuous random variables with joint PDF

$$f(x,y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1; 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a) Find the marginal PDF of $X$, $f(x)$, and $Y$, $f(y)$. *(5 points)*

b) Find the conditional PDF of $Y$ given $X$, $f(y|x)$. *(5 points)*

c) Find $P(Y \le 3/4 | X = 1/2)$. *(5 points)*

d) Find $E(Y|X = x)$. *(5 points)*

*Solution:*

a)

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^1 4xydy \\ &= 4x\frac{y^2}{2}\Big|_0^1 \\ &= 2x, \quad 0 \le x \le 1. \end{aligned}$$

Similarly, $f(y) = 2y$, $0 \le y \le 1$.

b)

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f(x)} \\ &= \frac{4xy}{2x} \\ &= 2y, \quad 0 \le y \le 1. \end{aligned}$$

c)

$$\begin{aligned} P(Y \le 3/4 | X = 1/2) &= \int_0^{3/4} f(y|x = 1/2)dy \\ &= \int_0^{3/4} 2ydy \quad \text{Using the conditional PDF in part b)} \\ &= y^2\Big|_0^{3/4} = \frac{9}{16}. \end{aligned}$$

d)

$$\begin{aligned} E(Y|X = x) &= \int_{-\infty}^{\infty} yf(y|x)dy \quad \text{def'n of conditional expectation} \\ &= \int_0^1 y(2y)dy = \frac{2y^3}{3}\Big|_0^1 = \frac{2}{3}. \end{aligned}$$

□

**Problem 6**
Let $X$ be a random variable with MGF

$$m(t) = \begin{cases} \frac{e^t - e^t}{2t}, & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

a) Give the distribution of $X$. *(10 points)*

b) Compute $E(X)$ and $V(X)$. *(10 points)*

*Solution:*

a) Matching the MGF above with known MGF formulas, we know that $X \sim U(-1, 1)$, where $\theta_1 = -1$ and $\theta_2 = 1$.

b) Using the mean and variance formula of a uniform RV, we have

$$\begin{aligned} E(X) &= \frac{\theta_1 + \theta_2}{2} = \frac{-1+1}{2} = 0 \\ V(X) &= \frac{(\theta_2 - \theta_1)^2}{12} = \frac{\{1-(-1)\}^2}{12} = \frac{4}{12} = \frac{1}{3}. \end{aligned}$$

□

**Problem 7**

Let $X$ and $Y$ be random variables such that

$$E(X) = 1, \quad E(X^2) = 3, \quad E(XY) = -4, \quad E(Y) = 2, \quad V(Y) = 25.$$

a) Find $E(2X + Y)$. *(4 points)*

b) Find $E\{X(2X + Y)\}$. *(4 points)*

c) Find $\text{Cov}(X, 2X + Y)$. *(4 points)*

d) Find $V(2X + Y)$. *(4 points)*

e) Find $\text{Corr}(X, 2X + Y)$. *(4 points)*

*Solution:*

a)

$$\begin{aligned} E(2X+Y) &= 2E(X) + E(Y) \quad \text{linearity of expectation} \\ &= 2(1) + 2 \quad \text{given} \\ &= 4. \end{aligned}$$

b)

$$\begin{aligned} E\{X(2X+Y)\} &= E(2X^2 + XY) \\ &= 2E(X^2) + E(XY) \quad \text{linearity of expectation} \\ &= 2(3) + (-4) \quad \text{given} \\ &= 2. \end{aligned}$$

c)

$$\begin{aligned} \text{Cov}(X, 2X+Y) &= E\{X(2X+Y)\} - E(X)E(2X+Y) \\ &\quad \text{def'n of covariance: } \text{Cov}(X,Y) = E(XY) - E(X)E(Y) \\ &= 2 - (1)(4) \quad \text{answers from part a) and b)} \\ &= -2. \end{aligned}$$

d)

$$\begin{aligned} V(2X+Y) &= 2^2V(X) + V(Y) + 2\text{Cov}(2X, Y) \\ &\quad \text{Variance of the sum: } V(X+Y) = V(X) + V(Y) + 2\text{Cov}(X,Y) \\ &= 4V(X) + V(Y) + 2(2)\text{Cov}(X,Y) \\ &\quad \text{Covariance of linear transform: } \text{Cov}(aX+b, cY+d) = ac\text{Cov}(X,Y). \text{ Here } a = 2, c = 1. \\ &= 4[E(X^2) - \{E(X)\}^2] + 25 + 4\{E(XY) - E(X)E(Y)\} \\ &\quad \text{def'n of variance and covariance} \\ &= 4(3 - 1^2) + 25 + 4(-4 - (1)(2)) \\ &= 9. \end{aligned}$$

e)

$$\begin{aligned}
\text{Corr}(X, 2X+Y) &= \frac{\text{Cov}(X, 2X+Y)}{\sqrt{V(X)V(2X+Y)}} && \text{def'n of correlation} \\
&= \frac{-2}{\sqrt{(2)(9)}} && \text{answers from part c) and d)} \\
&= \frac{-2}{\sqrt{18}} = -0.47.
\end{aligned}$$

□

Extra credit: Estimate your score for this midterm exam. If your score is within the range of $\pm 5$ points of your guess, you will get 5 extra points.