

STAT 3375Q: Introduction to Mathematical Statistics I
Spring 2024

Quiz 2 Review Exercises

Quiz Date: 7 February, 2024

Problem 1

Prove that for a random variable X with expected value $E(X) = \lambda$,

$$V(X) = E\{X(X - 1)\} + \lambda - \lambda^2.$$

Solution:

Approach 1.

$$\begin{aligned} V(X) &= E\{(X - \lambda)^2\} \\ &= E(X^2 - 2\lambda X + \lambda^2) \\ &= E(X^2 - X + X) - 2\lambda E(X) + \lambda^2 \\ &= E(X^2 - X) + E(X) - 2\lambda^2 + \lambda^2 \\ &= E(X^2 - X) + E(X) - 2\lambda^2 + \lambda^2 \\ &= E\{X(X - 1)\} + \lambda - \lambda^2 \end{aligned}$$

Approach 2.

$$\begin{aligned} V(X) &= E(X^2) - \lambda^2 \\ &= E(X^2) - \lambda + \lambda - \lambda^2 \\ &= E(X^2 - X) + \lambda - \lambda^2 \\ &= E\{X(X - 1)\} + \lambda - \lambda^2 \end{aligned}$$

□

Problem 2

A manufacturer is sending 10 boxes out for shipment today. Unfortunately, some of the boxes have defective items.

Box #	1	2	3	4	5	6	7	8	9	10
# of defective items	0	0	1	0	2	2	0	0	1	3

- One of these boxes is to be selected at random for shipment to a particular customer. Let X be the number of defective items in the selected box. What is the probability distribution of X ?
- What is the expected value of defective items?
- Another manufacturer is known to have X^2 defective items in each of the boxes numbered 1 to 10. If this manufacturer sends out a randomly selected box, what is the expected number of defective items the customer will receive?

Solution:

x	$p(x)$
0	$\frac{5}{10}$
1	$\frac{2}{10}$
2	$\frac{2}{10}$
3	$\frac{1}{10}$

a) (4 pts)

b) $E(X) = 0 \left(\frac{5}{10}\right) + 1 \left(\frac{2}{10}\right) + 2 \left(\frac{2}{10}\right) + 3 \left(\frac{1}{10}\right) = 0.9.$ (3 pts)

c) $E(X^2) = 0^2 \left(\frac{5}{10}\right) + 1^2 \left(\frac{2}{10}\right) + 2^2 \left(\frac{2}{10}\right) + 3^2 \left(\frac{1}{10}\right) = 1.9.$ (3 pts)

Problem 3

Suppose the random variable X takes on possible values $x = 0, 1, 2$ and has a probability mass function $f(x) = \frac{2x+3}{k}$, determine the value of k .

Solution:

x	$p(x)$
0	$3/k$
1	$5/k$
2	$7/k$

$$\frac{3}{k} + \frac{5}{k} + \frac{7}{k} = 1 \Rightarrow k = 15.$$

□

Problem 4

A box contains 5 red and 5 blue marbles. Two marbles are drawn randomly. If they are the same color, then you win \$1.10. If they are different colors, you lose \$1.00. Compute

- the expected value of the amount you win
- the variance of the amount you win

Solution:

Let X be a random variable for the amount you win. Let R be the event that a red marble is drawn and B be the event that a blue marble is drawn.

Event	x	$p(x)$
BB	1.1	$\binom{5}{10} \binom{4}{9} = \frac{2}{9}$ or $C_2^5/C_2^{10} = \frac{2}{9}$
RB \cup BR	-1	$\binom{5}{10} \binom{5}{9} + \binom{5}{10} \binom{5}{9} = \frac{5}{9}$
RR	1.1	$\binom{5}{10} \binom{4}{9} = \frac{2}{9}$ or $C_2^5/C_2^{10} = \frac{2}{9}$

- $E(X) = 1.1(2/9) + 1.1(2/9) - 1(5/9) = -0.6/9 = -0.067$.
- $E(X^2) = 1.1^2(2/9) + 1.1^2(2/9) + (-1)^2(5/9) = 9.84/9 = 1.093$
 $V(X) = E(X^2) - \{E(X)\}^2 = 1.093 - (-0.067)^2 = 1.089$.

Problem 5

If $E(X) = 1$ and $V(X) = 5$, find

a. $E\{(2 + X)^2\}$

b. $V(4 + 3X)$

Solution:

- a. $E\{(2 + X)^2\} = E(4 + 4X + X^2) = 4 + 4E(X) + E(X^2) = 4 + 4 + E(X^2) = 8 + E(X^2)$.
To solve for $E(X^2)$, we know $V(X) = E(X^2) - \{E(X)\}^2 \Rightarrow 5 = E(X^2) - 1^2 \Rightarrow E(X^2) = 6$.
Thus, $E\{(2 + X)^2\} = 8 + 6 = 14$.
- b. $V(4 + 3X) = 3^2V(X) = 45$.