UCONN UNIVERSITY OF CONNECTICUT

Department of Statistics
College of Liberal Arts & Sciences

STAT 3375Q: Introduction to Mathematical Statistics I

Spring 2024

Quiz 3 Review Exercises Quiz Date: 28 February 2024

Problem 1

Let X have a PDF defined by

$$f(x) = \begin{cases} cxe^{-x}, & x \geq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

a) Find the constant c.

b) Find $P(1 \leq X \leq 3)$.

Solution:

a)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ \Rightarrow 1 &= c\int_{2}^{\infty} xe^{-x}dx \\ \Rightarrow 1 &= c\left(-xe^{-x}\Big|_2^\infty - \int_2^\infty (-e^{-x})dx\right) \\ \Rightarrow 1 &= c\left(2e^{-2} - e^{-x}\Big|_2^\infty\right) \\ \Rightarrow 1 &= c\left(2e^{-2} + e^{-2}\right) \\ \Rightarrow 1 &= 3ce^{-2} \\ \Rightarrow c &= \frac{e^2}{3}. \end{aligned}$$

Integration by parts:

- $u = x \Rightarrow du = dx$
- $dv = e^{-x}dx \Rightarrow v = -e^{-x}$

Apply L'Hospital's rule to evaluate $\lim_{x\to\infty} -xe^{-x}$. Since $\lim_{x\to\infty} \frac{-x}{e^x} = \frac{-\infty}{\infty} \Rightarrow \lim_{x\to\infty} -xe^{-x} = \lim_{x\to\infty} \frac{-x}{e^x} = \lim_{x\to\infty} \frac{-1}{e^x} = 0$.

b)

$$\begin{aligned} P(1 \leq X \leq 3) &= \int_2^3 \frac{e^2}{3} xe^{-x}dx \\ &= \frac{e^2}{3}\left(-xe^{-x}\Big|_2^3 - \int_2^3 (-e^{-x})dx\right) \\ &= \frac{e^2}{3}\left(-3e^{-3} + 2e^{-2} - e^{-x}\Big|_2^3\right) \\ &= \frac{e^2}{3}\left(-3e^{-3} + 2e^{-2} - e^{-3} + e^{-2}\right) \\ &= \frac{e^2}{3}\left(-4e^{-3} + 3e^{-2}\right) \\ &= -\frac{4}{3}e^{-1} + 1 = 0.51. \end{aligned}$$

Problem 2
Let X denote a continuous random variable with PDF

$$f(x) = \begin{cases} \frac{x}{8}, & 0 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Define Y to be the integer that is closest X.

a) Explain why Y is a discrete random variable and give possible values for Y.

b) Compute the PMF of Y.

Solution:

a) Possible values of Y are 0, 1, 2, 3, and 4. Thus, Y is discrete.

b) To get the PMF, we need to compute the probabilities for each possible values of Y.

$$\begin{aligned}
P(Y=0) &= P(0 \le X < 0.5) = \int_0^{0.5} f(x)dx = \frac{x^2}{16}\Big|_0^{0.5} = \frac{1}{64}. \\
P(Y=1) &= P(0.5 \le X < 1.5) = \int_{0.5}^{1.5} f(x)dx = \frac{x^2}{16}\Big|_{0.5}^{1.5} = \frac{9}{64} - \frac{1}{64} = \frac{1}{8}. \\
P(Y=2) &= P(1.5 \le X < 2.5) = \int_{1.5}^{2.5} f(x)dx = \frac{x^2}{16}\Big|_{1.5}^{2.5} = \frac{25}{64} - \frac{9}{64} = \frac{1}{4}. \\
P(Y=3) &= P(2.5 \le X < 3.5) = \int_{2.5}^{3.5} f(x)dx = \frac{x^2}{16}\Big|_{2.5}^{3.5} = \frac{49}{64} - \frac{25}{64} = \frac{24}{64} = \frac{3}{8}. \\
P(Y=4) &= P(3.5 \le X < 4) = \int_{3.5}^{4} f(x)dx = \frac{x^2}{16}\Big|_{3.5}^{4} = 1 - \frac{49}{64} = \frac{15}{64}.
\end{aligned}$$

The PMF of Y is

$$p(y) = \begin{cases} \frac{1}{64}, & y = 0 \\ \frac{1}{8}, & y = 1 \\ \frac{1}{4}, & y = 2 \\ \frac{3}{8}, & y = 3 \\ \frac{15}{64}, & y = 4 \\ 0, & \text{elsewhere.} \end{cases}$$

Problem 3
Divide a stick into two parts. Find the probability that the larger part of the stick is at least three times the shorter.

Solution:
Assume the segment is in the interval $(0, 1)$ and let $X \sim U(0, 1)$.
Let X be a point in the interval $(0, 1)$ where we break the segment.
Two things can happen:

- Either $X \leq 1 - X \Rightarrow X \leq \frac{1}{2}$ (let this be event E_1)

- or $X > 1 - X \Rightarrow X > \frac{1}{2}$ (let this be event E_2).

  We want to find the probability that the longer segment is at least three times the shorter, i.e.,

$$P(1 - X > 3X|E_1)P(E_1) + P\{X > 3(1 - X)|E_2\}P(E_2).$$

Computing the values of each probabilities, we have

- $P(E_1) = P(X \leq \frac{1}{2}) = \frac{1}{2}$.
  Recall CDF of $U(a, b)$ is $F(x) = \frac{x-a}{b-a}$. Here, $x = \frac{1}{2}$, $a = 0$, and $b = 1$.
- $P(E_2) = P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$.
- $P(1 - X > 3X|E_1) = \frac{P\{(X<\frac{1}{4})\cap E_1\}}{P(E_1)} = \frac{P(X<\frac{1}{4})}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$.
- $P\{X > 3(1 - X)|E_2\} = \frac{P\{(X>\frac{3}{4})\cap E_2\}}{P(E_2)} = \frac{P(X>\frac{3}{4})}{\frac{1}{2}} = \frac{1-P(X\leq\frac{3}{4})}{\frac{1}{2}} = \frac{1-\frac{3}{4}}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$.

Hence,

$$P(1 - X > 3X|E_1)P(E_1) + P\{X > 3(1 - X)|E_2\}P(E_2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

□

Problem 4

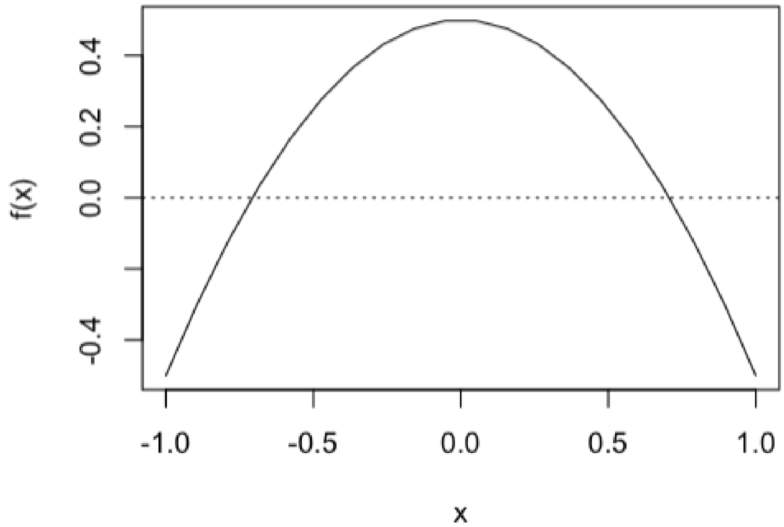
Suppose we have the following function:

$$f(y) = \begin{cases} c\left(\frac{1}{2} - y^2\right), & -1 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Is this a valid PDF? If not, is there a c for which this becomes a valid PDF?

Solution:

The $f(y)$ above will not make a valid PDF. When $c = 0$, then it does not integrate to 1. When $c \neq 0$, then there is an interval in $-1 \leq y \leq 1$ over which the integral is negative, and therefore does not represent a valid PDF over this interval.



Problem 5
Derive the PDF of $|X|$ where $X \sim U(-1, 1)$.

Solution:
Since X is uniform on the interval $(-1, 1)$, its PDF is:

$$f(x) = \begin{cases} \frac{1}{2}, & -1 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The PDF of $|X|$ can be derived from its CDF, $F_{|X|}(x)$. For $0 \leq x \leq 1$,

$$F_{|X|}(x) = P(|X| \leq x) = P(-x \leq X \leq x) = \int_{-x}^{x} \frac{1}{2} dt = \frac{1}{2} t \Big|_{-x}^{x} = \frac{1}{2}(2x) = x.$$

We can now get the PDF of $|X|$ from the CDF as follows:

$$f_{|X|}(x) = \frac{d}{dx} F_{|X|}(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$