Department of Statistics
College of Liberal Arts & Sciences

STAT 3375Q: Introduction to Mathematical Statistics I

Spring 2024

Quiz 4 Review Exercises Quiz Date: 20 March 2024

Problem 1
Suppose the grades on this quiz is normally distributed with a mean score of 70 points and standard deviation of 10 points. Furthermore, suppose I decide to give the top 10% a bonus of 5 points. What should be the cutoff score to merit the bonus points?

Solution:

- We want to find the cutoff score x such that

$$P(X \geq x) = 0.10.$$

- This is equivalent to finding the cutoff score x such that

$$\begin{aligned} P\left(\frac{X-\mu}{\sigma} \geq \frac{x-\mu}{\sigma}\right) &= 0.10 \\ \Rightarrow P\left(Z \geq \frac{x-\mu}{\sigma}\right) &= 0.10 \\ \Rightarrow 1 - P\left(Z < \frac{x-\mu}{\sigma}\right) &= 0.10 \\ \Rightarrow P\left(Z < \frac{x-\mu}{\sigma}\right) &= 0.90. \end{aligned}$$

  From the Z-table, $P(Z < 1.28) \approx 0.9$. This means that

$$\frac{x-\mu}{\sigma} = 1.28.$$

  Solving for x and replacing $\mu = 70$ and $\sigma = 10$ (given), we have

$$\begin{aligned} \frac{x-70}{10} &= 1.28 \\ \Rightarrow x - 70 &= 12.8 \\ \Rightarrow x &= 82.8. \end{aligned}$$

- Thus, the cutoff score for the bonus points is 82.8. □

Problem 2

Let X have MGF given by

$$m(t) = \frac{1}{3}e^t + \frac{2}{3}e^{2t}, \quad t \in \mathbb{R}.$$

a) What is the distribution of X?

b) Find the expected value and variance of X.

Solution:

a) Matching the MGF above to the MGF formula $m(t) = E(e^{tX}) = \sum_y e^{tx}p(x)$, we know that the MGF above corresponds to a discrete random variable with PMF:

$$p(x) = \begin{cases} \frac{1}{3}, & \text{if } x = 1, \\ \frac{2}{3}, & \text{if } x = 2, \\ 0, & \text{elsewhere.} \end{cases}$$

b)

$$\begin{aligned} m'(t) &= \frac{1}{3}e^t + \frac{4}{3}e^{2t} \\ E(X) = m'(0) &= \frac{1}{3}e^{(0)} + \frac{4}{3}e^{2(0)} = \frac{5}{3}. \end{aligned}$$

$$\begin{aligned} m''(t) &= \frac{1}{3}e^t + \frac{8}{3}e^{2t} \\ E(X^2) = m''(0) &= \frac{1}{3}e^{(0)} + \frac{8}{3}e^{2(0)} = \frac{9}{3} = 3. \\ V(X) &= E(X^2) - \{E(X)\}^2 = 3 - \left(\frac{5}{3}\right)^2 = \frac{2}{9}. \end{aligned}$$

$\square$

Problem 3
Verify that the standard normal PDF

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty,$$

is a valid PDF.

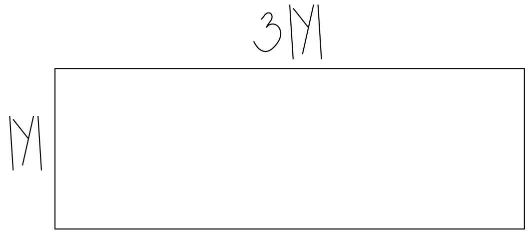
Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(z) dz &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= 2 \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \qquad \text{since } \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \text{ is an even function} \\
&= 2 \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} z^{-1} e^{-\frac{z^2}{2}} z dz \qquad \text{multiply a factor of 1} \\
&= 2 \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (\sqrt{2t})^{-1} e^{-t} dt \qquad \text{Let } t = z^2/2 \Rightarrow dt = zdz. \text{ Also, this implies that } z = \sqrt{2t}. \\
&= 2 \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{0}^{\infty} t^{-\frac{1}{2}} e^{-t} dt \\
&= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} t^{-\frac{1}{2}} e^{-t} dt \\
&= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{1}{2}-1} e^{-t} dt \\
&= \frac{1}{\sqrt{\pi}} \Gamma(1/2) \qquad \text{Recall the Gamma function: } \Gamma(\alpha) = \int_{0}^{\infty} y^{\alpha-1} e^{-y} dy. \\
&= \frac{1}{\sqrt{\pi}} \sqrt{\pi} \qquad \text{Recall properties of the Gamma function: } \Gamma(1/2) = \sqrt{\pi}. \\
&= 1.
\end{aligned}$$

Therefore, the standard normal PDF is a valid PDF. □

Problem 4

Suppose $Y \sim \mathcal{N}(\mu, \sigma^2)$. Find the expected value of the area of the rectangle below.



Solution:

- Given: $Y \sim \mathcal{N}(\mu, \sigma^2)$, $L = 3|Y|$, $W = |Y|$.
- Formula for area of rectangle: $A = L \times W$.
- Thus,

$$
\begin{aligned}
E(A) &= E(3|Y| \times |Y|) \\
&= 3E(Y^2) && \text{linearity property of expectation} \\
&= 3[V(Y) + \{E(Y)\}^2] && \text{variance formula} \\
&= 3(\sigma^2 + \mu^2). && \text{given}
\end{aligned}
$$

□

Problem 5
Suppose that X has the Gamma distribution with parameters α and β. Let c be a positive constant. Show that cX has the Gamma distribution with parameters α and $c\beta$.

Solution:
We can use the MGF to solve this problem. The MGF of cX is

$$
\begin{aligned}
m_{cX}(t) &= E\left(e^{tcX}\right) \\
&= E\left(e^{(tc)X}\right) \\
&= m_X(ct) \qquad \text{def'n of MGF} \\
&= \frac{1}{(1-\beta ct)^{\alpha}} \qquad \text{MGF of Gamma since } X \sim \text{Gam}(\alpha, \beta).
\end{aligned}
$$

The MGF above is identical to the MGF of a Gamma distribution with parameters α and $c\beta$. Thus, $cX \sim \text{Gam}(\alpha, c\beta)$.