Department of Statistics
College of Liberal Arts & Sciences

STAT 3375Q: Introduction to Mathematical Statistics I

Spring 2024

Quiz 5 Review Exercises Quiz Date: 17 April 2024

Problem 1
Let the random variable X have PDF $f(x) = \frac{30}{4}x^2(1-x)^2$ for $0 \leq x \leq 1$. Find the PDF of $Y = X^2$ using the Jacobian method.

Solution:

- Domain of X: $0 \leq x \leq 1$
- Codomain of Y: $0 \leq y \leq 1$
- Transformation: $h(x) = x^2$
- Inverse:
  Let $y = x^2$. To get the inverse, we need to solve for x. Solving for x, we have $x = \sqrt{y}$. Therefore, $h^{-1}(y) = \sqrt{y}$.
- Jacobian: $\frac{dh^{-1}(y)}{dy} = \frac{1}{2}y^{-1/2}$

$$
\begin{aligned}
f_Y(y) &= f_X\{h^{-1}(y)\}\left|\frac{dh^{-1}(y)}{dy}\right| \\
&= \frac{30}{4}(\sqrt{y})^2(1-\sqrt{y})^2\left|\frac{1}{2}y^{-1/2}\right| \\
&= \frac{30}{8\sqrt{y}}y(1-2\sqrt{y}+y) \\
&= \frac{15}{4}\sqrt{y}(1-2\sqrt{y}+y).
\end{aligned}
$$

□

Problem 2
Find a transformation $G(U)$ such that if U has a uniform distribution on $(0, 1)$, then $G(U)$ has a uniform distribution on $(2, 4)$.

Solution:

- Define the new RV: Let $X = G(U)$.
- By the probability integral transform theorem, the CDF of X is $G^{-1}(x)$.
- We want X to be uniform on $(2, 4)$. given
- This means that the CDF of X has the form $G^{-1}(x) = \frac{x-2}{2}$.

  Lec 14, Slide 4: CDF of $\mathcal{U}(a, b) : F(x) = \begin{cases} 0, & x < \theta_1 \\ \frac{x-\theta_1}{\theta_2-\theta_1}, & \theta_1 \leq x \leq \theta_2 \\ 1, & x > \theta_2. \end{cases}$
- To solve for the transformation $G(U)$, we need to find the inverse of $G^{-1}(x)$.
  - Let $u = \frac{x-2}{2}$.
  - Isolate x: $u = \frac{x-2}{2} \longrightarrow 2u = x - 2 \longrightarrow x = 2u + 2$.
  - Therefore, the required transformation of U is $G(U) = 2U + 2$. □

Problem 3
Let X_1 and X_2 have the joint PDF

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} e^{-(x_1+x_2)}, & \text{for } x_1 \geq 0, x_2 \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Consider two RVs U_1 and U_2 defined in the following manner:

$$U_1 = X_1 + X_2 \quad \text{and} \quad U_2 = \frac{X_1}{X_1+X_2}.$$

Find the joint PDF of U_1 and U_2.

Solution:

- Step 1: Identify the transformation. (new RVs = function of original RVs)
  $$\begin{cases} U_1 = X_1 + X_2 \\ U_2 = \frac{X_1}{X_1+X_2} \end{cases} \Rightarrow \begin{cases} u_1 = h_1(x_1,x_2) = x_1 + x_2 \\ u_2 = h_2(x_1,x_2) = \frac{x_1}{x_1+x_2} \end{cases}$$
  This means that if $x_1 \geq 0, x_2 \geq 0$, then $u_1 \geq 0$ and $0 \leq u_2 \leq 1$.

- Step 2: Deriving the inverse transformations, $h_1^{-1}(u_1,u_2)$ and $h_2^{-1}(u_1,u_2)$.
  (original RVs = function of new RVs)
  $$\begin{cases} U_1 = X_1 + X_2 \\ U_2 = \frac{X_1}{X_1+X_2} \end{cases} \Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ U_2 = \frac{X_1}{X_1+X_2} \end{cases} \Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ U_2 = \frac{U_1-X_2}{U_1-X_2+X_2} \end{cases}$$
  $$\Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ U_2 = \frac{U_1-X_2}{U_1} \end{cases} \Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ U_1U_2 = U_1 - X_2 \end{cases} \Rightarrow \begin{cases} X_1 = U_1 - X_2 \\ X_2 = U_1 - U_1U_2 \end{cases}$$
  $$\Rightarrow \begin{cases} X_1 = U_1 - (U_1 - U_1U_2) \\ X_2 = U_1 - U_1U_2 \end{cases} \Rightarrow \begin{cases} X_1 = U_1U_2 \\ X_2 = U_1 - U_1U_2. \end{cases}$$
  Thus, the inverse transformations are $\begin{cases} x_1 = h_1^{-1}(u_1,u_2) = u_1u_2 \\ x_2 = h_2^{-1}(u_1,u_2) = u_1 - u_1u_2. \end{cases}$

- Step 3: Obtain the Jacobian of the inverse transformations: $\begin{cases} h_1^{-1}(u_1,u_2) = u_1u_2 \\ h_2^{-1}(u_1,u_2) = u_1 - u_1u_2. \end{cases}$
  $$\begin{aligned} J &= \begin{vmatrix} \frac{\partial h_1^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial h_1^{-1}(u_1,u_2)}{\partial u_2} \\ \frac{\partial h_2^{-1}(u_1,u_2)}{\partial u_1} & \frac{\partial h_2^{-1}(u_1,u_2)}{\partial u_2} \end{vmatrix} = \begin{vmatrix} u_2 & u_1 \\ 1-u_2 & -u_1 \end{vmatrix} \\ &= \{-u_2u_1 - u_1(1-u_2)\} = -u_1. \end{aligned}$$
  Recall determinant of a matrix formula: $ad - bc$

- Step 4: Apply the formula
  - (Given) original joint PDF: $f_{X_1,X_2}(x_1,x_2) = \begin{cases} e^{-(x_1+x_2)}, & \text{for } x_1 \geq 0, x_2 \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$

$$\begin{aligned} f_{U_1,U_2}(u_1,u_2) &= f_{X_1,X_2}\{h_1^{-1}(u_1,u_2), h_2^{-1}(u_1,u_2)\}|J| \\ &= e^{-(u_1u_2+u_1-u_1u_2)}|-u_1| \\ &= u_1e^{-u_1}, \quad u_1 \geq 0, \quad 0 \leq u_2 \leq 1. \quad \square \end{aligned}$$

Problem 4
Let $Y_1, Y_2, \ldots, Y_{50} \sim \mathcal{U}(0,1)$. Find the mean and variance of the maximum RV, i.e., $E\{Y_{(50)}\}$ and $V\{Y_{(50)}\}$.

Solution:
In Lec 20, Slide 21, we found that

$$f_{Y_{(50)}}(y) = 50y^{49}, \quad y \in [0,1],$$

which is a Beta$(50,1)$ distribution, where $\alpha = 50$ and $\beta = 1$.

In Lec 14, Slide 5, we know that the mean and variance of a Beta RV are as follows:

$$\begin{aligned}
E\{Y_{(50)}\} &= \frac{\alpha}{\alpha+\beta} \\
&= \frac{50}{50+1} \\
&\approx 0.98 \\
V\{Y_{(50)}\} &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{(50)(1)}{(50+1)^2(50+1+1)} \\
&= \frac{50}{135252} \approx 0.0004.
\end{aligned}$$

Problem 5
Let $X_1, X_2, X_3, X_4 \overset{iid}{\sim} \mathcal{U}(0,5)$. Find the PDF of $X_{(3)}$.

Solution:

- CDF of $\mathcal{U}(0,5)$: $F(x) = \frac{x}{5}, \quad 0 \le x \le 5$
- PDF of $\mathcal{U}(0,1)$: $f(x) = \frac{1}{5}, \quad 0 \le x \le 5$
- $n = 4$
- $k = 3$
- Use the formula for the PDF of the kth order statistic:

$$\begin{aligned}
f_{X_{(3)}}(x) &= nf(x)\binom{n-1}{k-1} F(x)^{k-1}\{1-F(x)\}^{n-k} \quad \text{formula} \\
&= 4\left(\frac{1}{5}\right)\binom{4-1}{3-1}\left(\frac{x}{5}\right)^{3-1}\left(1-\frac{x}{5}\right)^{4-3} \\
&= \frac{4}{5}\binom{3}{2}\left(\frac{x}{5}\right)^{2}\left(1-\frac{x}{5}\right) \\
&= \frac{4}{5}(3)\left(\frac{x^2}{25}\right)\left(\frac{5-x}{5}\right) \\
&= \frac{12}{5}\left(\frac{x^2}{25}\right)\left(\frac{5-x}{5}\right) \\
&= \frac{12x^2(5-x)}{625} \\
&= \frac{60x^2-12x^3}{625}, \quad 0 \le x \le 5.
\end{aligned}$$

□