Solutions for Week9 Discussion Session

4.139

The MGF of a normally distributed random variable Y with mean μ and variance σ^2 is $m_Y(t) = E(e^{tY}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$.

We want to find the MGF of X = -3Y +4.
Hence, the mgf for X is:
$m_X(t) = E(e^{tX}) = E(e^{t(-3Y+4)}) = E(e^{-3tY+4t}) = e^{4t}E(e^{-3tY}) = e^{4t}m_Y(-3t)$
$= e^{4t}e^{-3\mu t + \frac{9\sigma^2 t^2}{2}} = e^{(-3\mu+4)t + \frac{9\sigma^2 t^2}{2}}$
Let, $\mu' = -3\mu+4; \sigma' = 3\sigma$. By the uniqueness of mgfs X follows a normal distribution with mean $\mu' = -3\mu+4$ and variance $\sigma'^2 = 9\sigma^2$.

4.141

For given $\theta_1 < \theta_2$, the MGF of a uniform random variable on the interval (θ_1, θ_2) will be:
$E(e^{tX}) = \int_{\theta_1}^{\theta_2} e^{tx}\frac{1}{\theta_2-\theta_1}dx = \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} e^{tx}dx = \frac{1}{t(\theta_2-\theta_1)}e^{tx}|_{\theta_1}^{\theta_2} = \frac{e^{t\theta_2}-e^{t\theta_1}}{t(\theta_2-\theta_1)}$
Hence, mgf of uniform(θ_1, θ_2) is $\frac{e^{t\theta_2}-e^{t\theta_1}}{t(\theta_2-\theta_1)}$.

4.142

Y is uniformly distributed on the interval (0,1) and that $a > 0$ is a constant.
a) The MGF of Y is for given $\theta_1 = 0; \theta_2 = 1$: $m_Y(t) = \frac{e^{t\theta_2}-e^{t\theta_1}}{\theta_2-\theta_1} = \frac{e^t-1}{t}$. [from the previous exercise]

(b) The MGF of $W = aY$ is $m_W(t) = m_Y(at)$; as W is a linear function of Y.
Hence, $m_W(t) = \frac{e^{at}-1}{at}$. Therefore, W follows uniform$(0, a)$.

(c) The MGF of $W = -aY$ is $m_W(t) = m_Y(-at)$; as W is a linear function of Y.
Hence, $m_W(t) = \frac{e^{-at}-1}{-at} = \frac{e^{-at}-1}{t(0-a)}$. Therefore, W follows uniform$(-a, 0)$.

(d) The MGF of $W = aY + b$ is $m_W(t) = e^{bt}m_Y(at)$; as W is a linear function of Y.
Hence, $m_W(t) = e^{bt}\frac{e^{at}-1}{at} = \frac{e^{(b+a)t}-e^{bt}}{t(b+a-b)}$. Therefore, W follows uniform$(b, b+a)$.

4.143

The MGF of a Gamma random variable(Y) is: $m(t) = (1-\beta t)^{-\alpha}$
We will find the mean and variance from mgf. For finding mean from mgf, we will take first derivative of m(t) with respect to t i.e. $m'(t)$. And, $E(Y) = m'(t=0)$.
Hence, $m'(t) = \alpha\beta(1-\beta t)^{-\alpha-1}$
$m'(t=0) = \alpha\beta = E(Y)$
For finding variance from mgf, we will take second order derivative of m(t) with respect to t i.e. $m''(t)$. And,

$E(Y^2) = m''(t=0)$. And, $Var(Y) = E(Y^2) - E^2(Y)$
Hence, $m''(t) = \alpha\beta^2(\alpha+1)(1-\beta t)^{-\alpha-2}$
$m''(t=0) = \alpha(\alpha+1)\beta^2 = E(Y^2)$
And, $Var(Y) = E(Y^2) - E^2(Y) = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$

4.181

Suppose Y is a normally distributed random variable with mean μ and variance σ^2.
The mgf is: $m_Y(t) = E(e^{tY}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$.

Now, $Z = \frac{Y-\mu}{\sigma} = \frac{1}{\sigma}Y + \frac{-\mu}{\sigma}$; which is a linear function of Y.

So, mgf of Z is: $e^{\frac{-\mu}{\sigma}t} m_Y(\frac{1}{\sigma}t) = e^{\frac{-\mu}{\sigma}t} e^{\frac{\mu t}{\sigma} + \frac{\sigma^2 t^2}{2\sigma^2}} = e^{\frac{t^2}{2}}$
By the uniqueness of mgfs Y follows a normal distribution with mean 0 and variance 1.