Solutions for Week10 Discussion Session

5.21

Given $P(y_1, y_2)$ is the pmf of i_1, y_2 for $0 \le y_1 \le 3; 0 \le y_2 \le 3; 1 \le y_1 + y_2 \le 3$.
(a) To find the marginal of y_1, we will sum it(given pmf) up with respect to y_2.
Hence, $\Sigma_{y_2=0}^{3} \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}$
And, the marginal of y_1 is hypergeometric with N=9, n= 3, r=4.

(b) Similarly to get the marginal of y_2, we will sum it(given pmf) up with respect to y_1. And, the marginal of y_2 is hypergeometric with N=9, n= 3, r=3.
$P(y_1 = 1|y_2 = 2) = \frac{P(y_1=1, y_2=2)}{P(y_2=2)} = \frac{\binom{4}{1}\binom{3}{2}\binom{2}{0}}{\binom{9}{3}} / \frac{\binom{3}{2}\binom{6}{1}}{\binom{9}{3}} = 2/3$

(c) If we let Y_3 denote the number of divorced executives among the three selected for promotion, then $Y_3 = 3 - Y_1 - Y_2$. So, for $y_3 = 1 \implies 1 = 3 - y_1 - 1 \implies y_1 = 1$
Hence, $P(y_3 = 1|y_2 = 1) = P(y_1 = 1|y_2 = 1) = \frac{P(y_1=1, y_2=1)}{P(y_2=1)} = \frac{\binom{3}{1}\binom{2}{1}\binom{4}{1}}{\binom{9}{3}} / \frac{\binom{3}{1}\binom{6}{2}}{\binom{9}{3}} = 8/15$

(d) The marginal distribution derived in (a) is the same we encountered in section 3.7, the hypergeometric distributions with N = 9, n =3, and r =4.

5.23

(a) Given, $f(y_1, y_2) = 3y_1; 0 < y_2 < y_1 < 1$
The marginal density function of y_2 is: $f(y_2) = \int_{y_2}^{1} 3y_1 dy_1 = \frac{3}{2} y_1^2|_{y_2}^{1} = 3/2(1 - y_2^2); 0 < y_2 < 1$

(b) For $y_2 > 0 \& y_2 < y_1 < 1$ the conditional density $f(y_1|y_2)$ is defined.

(c)First we have to find the conditional density of $f(y_2|y_1) = \frac{f(y_1, y_2)}{f(y_1)}$.
$f(y_1) = \int_0^{y_1} 3y_1 dy_2 = 3y_1 y_2|_0^{y_1} = 3y_1^2; 0 < y_1 < 1$
$f(y_2|y_1) = \frac{3y_1}{3y_1^2} = \frac{1}{y_1}; 0 < y_2 < y_1$. For a given value of y_1 the conditional density behaves like a Uniform$(0, y_1)$.
Now, $f(y_2 > 1/2|y_1 = 3/4)$ follows an Uniform$(0, 3/4)$. Hence, $P(y_2 > 1/2|y_1 = 3/4) = \frac{1}{3/4}(3/4 - 1/2) = 1/3$.

5.27

Given, $f(y_1, y_2) = 6(1 - y_2); 0 < y_1 < y_2 < 1$
(a) The marginals of $y_1 \& y_2$ are:
$f(y_1) = \int_{y_1}^{1} 6(1 - y_2) dy_2 = 3(1 - y_1)^2; 0 < y_1 < 1$
$f(y_2) = \int_0^{y_2} 6(1 - y_2) dy_1 = 6y_2(1 - y_2); 0 < y_2 < 1$

(b) $P(y_2 \leq 1/2 | y_1 \leq 3/4) = \frac{\int_0^{1/2}\int_0^{y_2} 6(1-y_2)dy_1 dy_2}{\int_0^{3/4} 3(1-y_1)^2 dy_1} = 32/63.$

(c) $f(y_1|y_2) = \frac{f(y_1,y_2)}{f(y_2)} = \frac{6(1-y_2)}{6y_2(1-y_2)} = \frac{1}{y_2}; 0 < y_1 < y_2$

(d) $f(y_2|y_1) = \frac{6(1-y_2)}{3(1-y_2)^2} = \frac{2(1-y_2)}{(1-y_1)^2}; 0 < y_1 < y_2 < 1$

(e) $P(y_2 \geq 3/4 | y_1 = 1/2) = \frac{\int_{3/4}^1 2(1-y_2)dy_2}{1/4} = \frac{1}{4}$

5.35

Given that, two minutes elapse between a customer's arrival at the store and his departure from the service window. That means, $y_1 = 2$. From exercise 5.33, we have, $f(y_1, y_2) = e^{-y_1}$; so, $f(y_2|y_1) = \frac{f(y_1,y_2)}{f(y_1)}$.
Hence, $f(y_1) = \int_0^{y_1} e^{-y_1} dy_2 = ye^{-y_1}; y_1 > 0$
Therefore, $f(y_2|y_1) = \frac{f(y_1,y_2)}{f(y_1)} = \frac{e^{-y_1}}{ye^{-y_1}} = \frac{1}{y_1}; 0 < y_2 < y_1$
That means the conditional density, $f(y_2|y_1 = 2)$ follows Uniform $(0, 2)$
Now, $P(y_2 < 1|y_1 = 2) = \frac{1}{2}\int_0^1 1dy_2 = 0.5$

5.41

Let, Y denotes the number of defective in a random selection of 3 items.
And, p is the proportion of defectives, which follows Uniform$(0, 1)$.

Given that, $P(Y = y|p) = \binom{3}{y} p^y (1-p)^{3-y}; y = 0, 1, 2, 3.$
To find the unconditional distribution, first we will find the joint distribution; $P(Y, p) = P(Y|p)f(p)$
Then to get the marginal of Y which is the unconditional distribution of Y, we will integrate over the whole range of p. So, $P(Y) = \int_0^1 P(Y, p)dp$
Therefore, $P(Y = 2) = \int_0^1 P(Y = 2, p)dp = \int_0^1 \binom{3}{2} p^2(1-p)dp = \frac{1}{4}$