Solutions for Week12 Discussion Session

6.15

Let U have a uniform distribution on (0, 1).
The distribution function for U is $F_U(u) = P(U < u) = u; 0 \leq u \leq 1$.
For a function G, we require G(U) = Y where Y has distribution function $F_Y(y) = 1 - e^{-y^2}; y \geq 0$.
Note that,
$F_Y(y) = P(Y \leq y) = P(G(U) \leq y) = P(U \leq G^{-1}(y)) = F(G^{-1}(y)) = u$.
So it must be true that $G^{-1}(y) = 1 - e^{-y^2} = u$
Hence, $G(u) = [-ln(1-u)]^{-1/2}$

Therefore, the random variable Y = [–ln(1-U)]–1/2 has distribution function $F_Y(y)$.

6.20

Y follows uniform (0,1).
(a) We have to find the distribution of $W = Y^2$. We will use the CDF method.
$F_W(w) = P(W \leq w) = P(Y^2 < w) = P(Y < \sqrt{w}) = \sqrt{w}$.
And the range for w is: $0 < y < 1 \implies 0 < y^2 < 1 \implies 0 < w < 1$.

(b) We have to find the distribution of $W = \sqrt{Y}$. We will use the CDF method.
$F_W(w) = P(W \leq w) = P(\sqrt{Y} < w) = P(Y < w^2) = w^2$
And the range for w is: $0 < y < 1 \implies 0 < \sqrt{y} < 1 \implies 0 < w < 1$.

6.23

Given, Y has pdf $f(y) = 2(1-y); 0 \leq y \leq 1$

(a) We have to find the distribution of $U_1 = 2Y - 1$. We will use the Jacobian method to find the distribution of U_1.
$U_1 = 2Y - 1 \implies Y = \frac{U_1+1}{2}$
Take the first derivative of Y with respect to U_1; $dy/du_1 = 1/2$.
Hence, the Jacobian; $|J| = |dy/du_1| = 1/2$.
$f(u_1) = |J| f_{u_1}(y) = \frac{1}{2} 2(1 - \frac{U_1+1}{2}) = \frac{1-u_1}{2}$
Now, for the range of u_1, we have $0 \leq y \leq 1 \implies 0 \leq 2y \leq 2 \implies -1 \leq 2y - 1 \leq 1 \implies -1 \leq u_1 \leq 1$.
Hence, $f(u_1) = \frac{1-u_1}{2}; -1 \leq u_1 \leq 1$.

(b) We have to find the distribution of $U_2 = 1 - 2Y$. We will use the Jacobian method to find the distribution of U_2.
$U_2 = 1 - 2Y \implies Y = \frac{1-U_2}{2}$
Take the first derivative of Y with respect to U_2; $dy/du_2 = -1/2$.
Hence, the Jacobian; $|J| = |dy/du_2| = 1/2$.
$f(u_2) = |J| f_{u_2}(y) = \frac{1}{2} 2(1 - \frac{1-U_2}{2}) = \frac{1+u_2}{2}$
Now, for the range of u_2, we have $0 \leq y \leq 1 \implies 0 \leq 2y \leq 2 \implies 0 \geq -2y \geq -2 \implies 1 \geq 1 - 2y \geq$

$-1 \implies -1 \le u_2 \le 1$.
Hence, $f(u_2) = \frac{1+u_2}{2}; -1 \le u_2 \le 1$.

(c) Again, to find the distribution of $U_3 = Y^2$. We will use the Jacobian method to find the distribution of U_3.
$U_3 = Y^2 \implies Y = \sqrt{U_3}$
Take the first derivative of Y with respect to U_3; $dy/du_3 = \frac{1}{2\sqrt{u_3}}$.
Hence, the Jacobian; $|J| = |dy/du_3| = \frac{1}{2\sqrt{u_3}}$.
$f(u_3) = |J| f_{u_3}(y) = \frac{1}{2\sqrt{u_3}} 2(1-\sqrt{u_3}) = \frac{1-\sqrt{u_3}}{\sqrt{u_3}}$
Now, for the range of u_3, we have $0 \le y \le 1 \implies 0 \le y^2 \le 1 \implies 0 \le u_3 \le 1$.
Hence, $f(u_3) = \frac{1-\sqrt{u_3}}{\sqrt{u_3}}; 0 \le u_3 \le 1$.

6.28

Given, Y follows Uniform(0,1). So, $f(y) = 1; 0 < y < 1$
We have to find the distribution of $U = -2ln(Y)$. We will use the Jacobian method to find the distribution of U.
$U = -2ln(Y) \implies -U/2 = lnY \implies Y = exp(-U/2)$
Take the first derivative of Y with respect to U; $dy/du = exp(-u/2)(-1/2)$.
Hence, the Jacobian; $|J| = |dy/du_1| = exp(-u/2)(1/2)$.
$f(u) = |J| f_u(y) = exp(-u/2)(1/2).1 = exp(-u/2)(1/2)$
Now, for the range of u, we have $-\infty < lny < 0 \implies 0 < -2lny < \infty \implies 0 < u < \infty$.[as log is a decreasing function; so as $y \to 0 \text{then}, ln(y) \to \infty$]
Hence, $f(u) = \frac{1}{2} exp(-u/2); 0 < u < \infty$.
It is clear that U is now an exponential distribution with mean 2.

6.46

Given that, Y has a Gamma distribution with $\alpha = n/2$ for some positive integer n and β equal to some specified value.
Here we will use method of MGFs to find the distribution of $W = 2Y/\beta$.
We know the MGF of Y is: $M_Y(t) = E(e^{tY} = (1-\beta t)^{-\alpha}$
Now, to find the MGF of W is:

$$\begin{aligned}
M_W(t) &= E(e^{tW}) \\
&= E(e^{t(2Y/\beta)}) \\
&= E(e^{t(2Y/\beta)}) \\
&= E(e^{t'Y}); \text{where,} t' = 2t/\beta \\
&= (1-\beta t')^{-\alpha} \\
&= (1-\beta(2t/\beta))^{-\alpha} \\
&= (1-2t)^{-\alpha}
\end{aligned}$$

As $\alpha = n/2$, so, $M_W(t) = (1-2t)^{-\alpha} = (1-2t)^{-n/2}$. Which is a MGF of Chi-square distribution. Hence, $W = 2Y/\beta$ has a χ^2 distribution with n degrees of freedom.