# Solutions for Week13 Discussion Session

## 6.72

Given, $Y_1$ and $Y_2$ are independent and uniformly distributed over the interval (0,1). So,$F(y) = P(Y < y) = y; 0 < y < 1$.
(a) We have to find the PDF of $U_1 = min(Y_1, Y_2)$. We will use the CDF method. Hence,

$$\begin{aligned} F_{U_1}(u) &= P(U_1 < u) \\ &= P(min(Y_1, Y_2)) < u) \\ &= 1 - P(min(Y_1, Y_2)) > u) \\ &= 1 - P(Y_1 > u)P(Y_2 > u); \text{as they are independent} \\ &= 1 - (1-u)^2 \end{aligned}$$

We have $F_{U_1}(u) = 1 - (1-u)^2; 0 < u < 1$
$\implies \frac{dF_{U_1}(u)}{du} = f(u) = 2(1-u); 0 < u < 1.$
So, the pdf of $U_1$ is: $f(u) = 2(1-u); 0 < u < 1$. This is a $\beta(1,2)$ distribution.

(b) As f(u) is a beta distribution, hence, the $E(U_1) = \frac{\alpha}{\alpha+\beta} = \frac{1}{3}$.
The $Var(U_1) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{18}$.

## 6.73

Given, $Y_1$ and $Y_2$ are independent and uniformly distributed over the interval (0,1). So,$F(y) = P(Y < y) = y; 0 < y < 1$.
(a) We have to find the PDF of $U_2 = max(Y_1, Y_2)$. We will use the CDF method. Hence,

$$\begin{aligned} F_{U_2}(u) &= P(U_2 < u) \\ &= P(max(Y_1, Y_2)) < u) \\ &= P(Y_1 < u)P(Y_2 < u); \text{as they are independent} \\ &= (u)^2 \end{aligned}$$

We have $F_{U_2}(u) = (u)^2; 0 < u < 1$
$\implies \frac{dF_{U_2}(u)}{du} = f(u) = 2u; 0 < u < 1.$
So, the pdf of $U_2$ is: $f(u) = 2u; 0 < u < 1$. This is a $\beta(2,1)$ distribution.

(b) As f(u) is a beta distribution, hence, the $E(U_2) = \frac{\alpha}{\alpha+\beta} = \frac{2}{3}$.
The $Var(U_2) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{18}$.

## 6.74

Given, $Y_1, Y_2, ..., Y_n$ are independent and uniformly distributed over the interval (0,$\theta$). So, the cdf of $Y_i$ are: $F(y) = P(Y < y) = \frac{y}{\theta}; 0 < y < \theta$.

(a) We have to find the PDF of $Y_{(n)} = max(Y_1, Y_2, ..., Y_n)$. We will use the CDF method. Hence,

$$\begin{aligned} F_{Y_{(n)}}(y) &= P(Y_{(n)} < y) \\ &= P(max(Y_1, Y_2, ..., Y_n)) < y) \\ &= P(Y_1 < y)P(Y_2 < y)...P(Y_n < y); \text{as they are independent} \\ &= (\frac{y}{\theta})^n \end{aligned}$$

We have $F_{Y_{(n)}}(y) = (\frac{y}{\theta})^n; 0 < y < \theta$

(b) Now the pdf of $Y_{(n)}$ will be: $f_{Y_{(n)}}(y) = \frac{dF_{Y_{(n)}}(y)}{dy} = n(\frac{y}{\theta})^{n-1}\frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}; 0 < y < \theta.$

(c) $E(Y_{(n)}) = \int_0^\theta y\frac{ny^{n-1}}{\theta^n}dy = \int_0^\theta \frac{ny^n}{\theta^n}dy = \frac{ny^{n+1}}{(n+1)\theta^n}|_0^\theta = \frac{n\theta}{n+1}$
To find the variance we first calculate $E(Y_{(n)}^2)$, then $Var(Y_{(n)}) = E(Y_{(n)}^2) - [E(Y_{(n)})]^2$.
$E(Y_{(n)}^2) = \int_0^\theta y^2\frac{ny^{n-1}}{\theta^n}dy = \int_0^\theta \frac{ny^{n+1}}{\theta^n}dy = \frac{ny^{n+2}}{(n+2)\theta^n}|_0^\theta = \frac{n\theta^2}{n+2}$.
So, $Var(Y_{(n)}) = \frac{n\theta^2}{(n+1)^2(n+2)}$.

## 6.75

Given that the number of minutes that you need to wait for a bus is uniformly distributed on the interval [0,15].
If you take the bus five times, that means we have five observations for waiting time in a bus stop $Y_1, Y_2, Y_3, Y_, Y_5$ respectively. We have to find the probability that longest wait is less than 10 minutes i.e. $max(Y_1, Y_2, Y_3, Y_, Y_5) < 10$.

Therefore, $P(max(Y_1, Y_2, Y_3, Y_, Y_5) < 10) = P(Y_{(5)} < 10) = (\frac{10}{15})^5$; using the distribution function of $F_{Y_{(n)}}(y)$ from previous problem.
Hence, required probability is $P(Y_{(5)} < 10) = (10/15)^5 = .1317$.

## 6.80

Given that, $Y_1, Y_2, ..., Y_n$ be independent random variables, each with a beta distribution, with $\alpha = \beta = 2$.
So,the PDF of Y is $f(y) = 6y(1-y); 0 < y < 1$ and the CDF is:$F(y) = 3y^2 - 2y^3; 0 < y < 1$.
(a) CDF of $Y_{(n)}$ is:

$$\begin{aligned} F_{Y_{(n)}}(y) &= P(Y_{(n)} < y) \\ &= P(max(Y_1, Y_2, ..., Y_n)) < y) \\ &= P(Y_1 < y)P(Y_2 < y)...P(Y_n < y); \text{as they are independent} \\ &= (3y^2 - 2y^3)^n; 0 < y < 1 \end{aligned}$$

(b) The PDF is: $f_{Y_{(n)}}(y) = \frac{dF_{Y_{(n)}}(y)}{dy} = 6ny(1-y)(3y^2 - 2y^3)^{n-1}; 0 < y < 1$

(c)We have n=2. So,from the above expression, $f_{Y_{(2)}}(y) = 12y(1-y)(3y^2 - 2y^3); 0 < y < 1$
$E(Y_{(2)}) = \int_0^1 yf_{Y_{(2)}}(y)dy = \int_0^1 12y^2(1-y)(3y^2 - 2y^3)dy = 0.6286$