Solutions for Week6 Discussion Session

4.5

Y is a random variable that takes on only integer values 1,2,... and has distribution function $F(y)$.
From the definition of a distribution function we know that, $F(y) = P(Y \leq y)$
So, for $y = 1$, $P(Y \leq 1) = P(y = 1) \implies F(1) = P(y = 1)$
Now, for $y = 2$, $P(Y \leq 2) = P(y = 1) + P(y = 2)$; as Y takes the discrete values.
$\implies F(2) = F(1) + P(y = 2)$
$\implies F(2) - F(1) = P(y = 2)$
Similarly, for $y = 3$, $P(Y \leq 3) = P(y = 1) + P(y = 2) + P(y = 3)$
$\implies F(3) = F(2) + P(y = 3)$
$\implies F(3) - F(2) = P(y = 3)$
Therefore, we can generalize the expression for y=2,3,..

$$\begin{aligned} p(y) &= F(1); y = 1 \\ &= F(y) - F(y-1); y = 2, 3, .. \end{aligned}$$

4.9

(a) Here, set of possible values of Y represents a countable set. So, it is a discrete random variable.

(b) The values of Y are 2, 2.5,4,5.5,6,7 are assigned positive probabilities; as they are the cut-off points of the step diagram.

(c) The probability function for Y is by the definition is: $p(y) = F(y) - F(y-1)$
Here, y takes the values 2, 2.5,4,5.5,6 and 7. So,
$p(2) = F(2) = 1/8$
$p(2.5) = F(2.5) - F(2) = 1/8 - 3/16 = 1/16$
$p(4) = F(4) - F(2.5) = 1/2 - 3/16 = 5/16$
$p(5.5) = F(5.5) - F(4) = 5/8 - 1/2 = 1/8$
$p(6) = F(6) - F(5.5) = 11/16 - 5/8 = 1/16$
$p(7) = F(7) - F(6) = 1 - 11/16 = 5/16.$

(d) Median is the point which divides the whole data into two equal parts. So, upto the median value the probability will be 0.50. From the above probability distribution(discrete) of y, we get $p(2)+p(2.5)+p(4) = 0.50$. Hence y=4 is the median.

4.13

We are given that,
$$f(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 1, & 1 < y \leq 1.5 \\ 0, & \text{elsewhere} \end{cases}$$
(a) We have to find the F(y). By definition $F(y) = P(Y \leq y) = \int f(y)dy$; as this is a continuous case.

Therefore, $F(t) = \int_0^y tdt = \frac{y^2}{2}$; for $0 < y < 1$
Again, for $1 < y < 1.5; F(t) = \int_0^1 tdt + \int_1^y dt = \frac{1}{2} + (y-1)$

$$F(y) = \begin{cases} \frac{y^2}{2}, & 0 \le y \le 1 \\ y - \frac{1}{2}, & 1 < y \le 1.5 \\ 0, & \text{elsewhere} \end{cases}$$

(b) To find the probability of $P(0 \le Y \le 0.5)$; we use the formula $P(a \le Y \le b) = F(b) - F(a)$
Hence, for $0 < y < 0.5; F(y) = \frac{y^2}{2}$; So,$P(0 \le Y \le 0.5) = F(0.5) - F(0) = \frac{0.5^2}{2} - 0 = \frac{1}{8}$

(c) Similarly, for $0.5 < y < 1; F(y) = \frac{y^2}{2}$ and for $1 < y < 1.2, F(y) = y - 1$. Hence, $P(0.5 \le Y \le 1.2) = F(1.2) - F(0.5) = (1.2-1) - \frac{0.5^2}{2} = 0.575$

4.15

Given that,

$$f(y) = \begin{cases} \frac{b}{y^2}, & y > b \\ 0, & \text{elsewhere} \end{cases}$$; where b is the minimum possible time.

(a) For any $b > 0, f(y) \ge 0$
And, $\int_b^\infty f(y)dy = \int_b^\infty \frac{b}{y^2}dy = \frac{-b}{y}|_b^\infty = 1$
Hence, f(y) has the properties of a density function.

(b) For $b < y < \infty, F(y) = \int_b^y f(t)dt = \int_b^y \frac{b}{t^2}dt = \frac{-b}{t}|_b^y = \frac{-b}{y} + 1$

$$F(y) = \begin{cases} \frac{-b}{y} + 1, & y \ge b \\ 0, & \text{elsewhere} \end{cases}$$

(c) $P(Y > b + c) = 1 - P(Y \le b + c) = 1 - F(b+c) = 1 - \frac{-b}{b+c} - 1 = \frac{b}{b+c}$

(d) $P(Y > b + d|Y > b + c) = \frac{P(Y>b+d \cap Y>b+c)}{Y>b+c}$
As, d,c both positive numbers and $d > c$ hence some $x > b + d \implies x > b + c$. So, the intersection part is equal to $Y > b + d$.
$\frac{P(Y>b+d \cap Y>b+c)}{P(Y>b+c)} = \frac{P(Y>b+d)}{P(Y>b+c)} = \frac{1-P(Y \le b+d)}{1-P(Y \le b+c)} = \frac{1-F(b+d)}{1-F(b+c)} = \frac{1-\frac{-b}{b+d}-1}{1-\frac{-b}{b+c}-1} = \frac{b+c}{b+d}$

4.17

Given, $f(y) = \begin{cases} cy^2 + y, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$
(a) As it is a density function, $\int f(y)dy = 1$
$\int_0^1 (cy^2 + y)dy = 1$
$\implies c\frac{y^3}{3}|_0^1 + \frac{y^2}{2}|_0^1 = 1$
$\implies \frac{c}{3} + \frac{1}{2} = 1$
$\implies c = \frac{3}{2}$

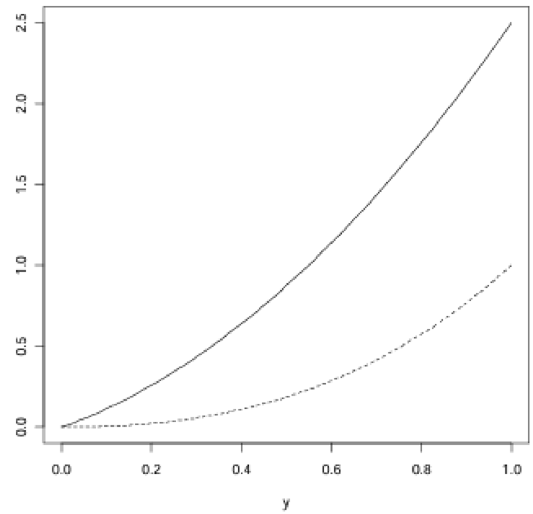

Figure 1: Graph

$$f(y) = \begin{cases} \frac{3}{2}y^2 + y, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

(b) $F(y) = \int_0^1 (ct^2 + t)dt = \frac{y^3+y^2}{2}$;for $0 < y < 1$

$$F(y) = \begin{cases} \frac{y^3+y^2}{2}, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

(c) The solid line graph for f(y) and the dotted line for F(y) in figure1.

(d)From part(b) we have, $F(0) = 0; F(-1) = 0$ as the points are not within (0,1). And $F(1) = P(Y < 1) = 1$, as it's a pdf.

(e) The probability that a randomly selected student will finish in less than half an hour i.e.
$P(Y < 0.5) = F(0.05) = \frac{0.5^3+0.5^2}{2} = 3/16$

(f) $P(Y > 0.5|Y > 0.25) = \frac{P(Y>0.5 \cap Y>0.25)}{P(Y>0.25} = \frac{P(Y>0.5)}{P(Y>0.25)} = \frac{104}{123}$